

# Delay-dependent robust output feedback stabilization of uncertain state-delayed systems with saturating actuators

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**Abstract.** The robust output feedback stabilization problem for state-delayed systems with time-varying delays and saturating actuators is addressed here. The systems considered are continuous-time, with parametric uncertainties entering all the matrices in the system representation. A saturating control law is designed and a region of initial conditions is specified within which local asymptotic stability of the closed-loop system is ensured. The designed controller is dependent on the time-delay and its rate of change. The controller is constructed in terms of the solution to a set of matrix inequalities.

## 1. Introduction

In recent years, much work has been devoted to the analysis and synthesis of controllers for state-delayed systems with or without parametric uncertainties; see [1] for earlier work. To the authors best knowledge, the issue of output feedback stabilization of uncertain state-delayed systems with saturating control was only addressed in [2].

In this context, the contribution of this paper can be summarized as follows:

- A delay-dependent robust output feedback stabilization problem is addressed for state-delayed systems in full generality, including actuators constraints and norm-bounded parametric uncertainties entering all the matrices in the system representation.

- In attempt to make the approach the least conservative, unlike in [2], the most efficient descriptor model transformation and Lyapunov-Krasovskii functional are employed, as in [3].

- The least conservative model for the actuator saturation; see [4], as delivered by the differential inclusions approach is employed here.

- The value of the time-delay as well as its rate of change are taken into account in the design method presented and further permit to reduce the conservativeness of the approach.

- The method developed here is applied to an example system which was previously used in [5], and shows that the new design yields less conservative results, in that stabilization is ensured for a larger set of initial conditions.

## 2. Problem statement

Consider the following uncertain time-delay system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u_{sat}(t), \\ y(t) &= C(t)x(t) + D(t)u_{sat}(t), \quad u_{sat}(t) = sat(u(t)), \\ sat(u(t)) &= [sat(u_1(t)) \quad sat(u_2(t)) \quad \dots \quad sat(u_m(t))] \end{aligned}$$

$$x(t_0 + \psi) = \phi(\psi), \quad \forall \psi \in [-h_{\max}, 0], \quad (t_0, \phi) \in R^+ \times C_{h_{\max}, n}^w \quad (1)$$

where  $x(t) \in R^n$  is the state vector,  $y(t) \in R^r$  is the output vector,  $u(t) \in R^m$  is the control input vector to the actuator (generated from the designed controller),  $u_{sat}(t) \in R^m$  is the control input vector to the plant. The time-delay  $h$  is a function of time and is assumed to be continuously differentiable, with its amplitude and rate of change bounded as follows:

$$0 \leq h(t) \leq h_{\max}, \quad 0 \leq \dot{h}(t) \leq \beta < 1, \quad \text{for all } t \geq 0 \quad (2)$$

where  $h_{\max}$  and  $\alpha$  are given positive constants. The uncertain matrices  $A(t)$ ,  $A_h(t)$ ,  $B(t)$ ,  $C(t)$  and  $D(t)$  are:

$$\begin{aligned} A(t) &= A + \Delta A(t), \quad A_h(t) = A_h + \Delta A_h(t), \quad B(t) = B + \Delta B(t), \\ C(t) &= C + \Delta C(t), \quad D(t) = D + \Delta D(t) \end{aligned} \quad (3)$$

where the matrices  $A$ ,  $A_h$ ,  $B$ ,  $C$  and  $D$  are real, and are assumed to be known exactly. The matrices  $\Delta A(t)$ ,  $\Delta A_h(t)$ ,  $\Delta B(t)$ ,  $\Delta C(t)$  and  $\Delta D(t)$  are real-valued, represent the norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$\begin{aligned} \begin{bmatrix} \Delta A(t) & \Delta B(t) \\ \Delta C(t) & \Delta D(t) \end{bmatrix} &= \begin{bmatrix} H_A \\ H_C \end{bmatrix} F(t) \begin{bmatrix} E_A & E_B \end{bmatrix}, \\ \Delta A_h(t) &= H_h F_h(t) E_h \end{aligned} \quad (4)$$

where  $F(t) \in R^{i \times j}$  and  $F_h(t) \in R^{i_h \times j_h}$  are real, uncertain and time-varying matrices with Lebesgue measurable entries which, additionally, meet the following requirements:  $F(t)F^T(t) \leq I$  and  $F_h(t)F_h^T(t) \leq I$ . The matrices  $H_A$ ,  $H_C$ ,  $H_h$ ,  $E_A$ ,  $E_B$  and  $E_h$  are known and real, and characterize the way in which the uncertain parameters in  $F(t)$  and  $F_h(t)$  enter the nominal matrices  $A$ ,  $A_h$ ,  $B$ ,  $C$  and  $D$ .

Finally, the following is assumed to hold:

**Assumption 1:**  $(A + A_h, B)$  is stabilizable and  $(C, A)$  is detectable.

**Assumption 2:** The input vector is subject to amplitude constraints, i.e.  $u \in U_0 \subset R^m$ , with

$$U_0 \triangleq \{u \in R^m; -\bar{u}_i \leq u_i \leq \bar{u}_i, \bar{u}_i > 0, i = 1 \dots m\}. \quad (5)$$

The results to be presented are concerned with providing sufficient conditions for the design of an observer-based dynamic output feedback law for system (1). This law is assumed to take the following form:

$$u(t) = K\hat{x}(t) \quad (6)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{sat}(t) + L(y(t) - C\hat{x}(t) - Du_{sat}(t)) \quad (7)$$

where  $\hat{x}(t) \in R^n$  is the observer state vector,  $K \in R^{m \times n}$  and  $L \in R^{n \times r}$  are the constant controller and observer gains, respectively. Using (1), (4) and (7), permits to write,

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} &= \left\{ \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix} + \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) [E_A \quad 0] \right\} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \\ &+ \left\{ \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H_h \\ 0 \end{bmatrix} F_h(t) [E_h \quad 0] \right\} \begin{bmatrix} x(t-h(t)) \\ \hat{x}(t-h(t)) \end{bmatrix} \\ &+ \left\{ \begin{bmatrix} B \\ B \end{bmatrix} + \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) E_B \right\} u_{sat}(t) \end{aligned} \quad (8)$$

Finally, introducing the following definitions,

$$\begin{aligned} \xi(t) &\triangleq \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \hat{A} \triangleq \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix}, \hat{A}_h \triangleq \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix}, \\ \Delta \hat{A}(t) &\triangleq \hat{H}F(t)\hat{E} \triangleq \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) [E_A \quad 0], \hat{B} \triangleq \begin{bmatrix} B \\ B \end{bmatrix}, \\ \Delta \hat{A}_h(t) &\triangleq \hat{H}_h F_h(t) \hat{E}_h \triangleq \begin{bmatrix} H_h \\ 0 \end{bmatrix} F_h(t) [E_h \quad 0], \\ \Delta \hat{B}(t) &\triangleq \hat{H}F(t)E_B \triangleq \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) E_B, \hat{A}(t) \triangleq \hat{A} + \Delta \hat{A}(t), \\ \hat{A}_h(t) &\triangleq \hat{A}_h + \Delta \hat{A}_h(t), \hat{B}(t) \triangleq \hat{B} + \Delta \hat{B}(t), \hat{K} = [0 \quad K] \end{aligned} \quad (9)$$

permits to re-write equation (8) in yet more aggregated compact form,

$$\dot{\xi}(t) = \hat{A}(t)\xi(t) + \hat{A}_h(t)\xi(t-h(t)) + \hat{B}(t)u_{sat}(t),$$

$$\text{with } u_{sat}(t) = \text{sat}(\hat{K}\xi(t)). \quad (10)$$

### The robust output feedback stabilization problem with saturating actuators:

Find a matrix  $K \in R^{m \times n}$ , a matrix  $L \in R^{n \times r}$  and a set of initial conditions  $S_0 \subset R^{2n}$  such that the closed-loop system (10) is asymptotically stable.

### 3. Main result

The following theorem provides sufficient conditions for robust output feedback stabilization of the uncertain time-delay system (10) with control saturation.

**Theorem 1.** Consider the system (10). Suppose that there exist,  $n \times n$ -matrices:  $P_{i11}, P_{i12}, P_{i21}, P_{i22}; i=1, \dots, 3$ ,  $W_{i11}, W_{i12}, W_{i21}, W_{i22}; i=1, \dots, 4$ ,  $R_{i11}, R_{i12}, R_{i21}, R_{i22}; i=1, \dots, 3$ ,  $S = S^T > 0$ , an  $m \times n$  matrix  $K$  and an  $n \times r$  matrix  $L$ , and a vector  $\underline{\alpha} \in R^m$ , and a positive scalar  $\gamma$ , which together with some suitable positive scalars  $\varepsilon_i; i=1, \dots, 3$ , satisfy the following matrix conditions:

$$\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix} > 0, P_{111} = P_{111}^T, P_{112} = P_{121}^T, P_{122} = P_{122}^T \quad (11)$$

$$\begin{bmatrix} R_{311} & R_{312} \\ R_{321} & R_{322} \end{bmatrix} > 0, R_{311} = R_{311}^T, R_{312} = R_{321}^T, R_{322} = R_{322}^T \quad (12)$$

$$\begin{bmatrix} R_{111} & R_{112} & R_{211} & R_{212} \\ R_{121} & R_{122} & R_{221} & R_{222} \\ R_{211}^T & R_{221}^T & R_{311} & R_{312} \\ R_{212}^T & R_{222}^T & R_{321} & R_{322} \end{bmatrix} > 0, R_{111} = R_{111}^T, R_{112} = R_{121}^T, \quad (13)$$

$$\begin{bmatrix} R_{122} & R_{122}^T \\ \Omega_{1,j} & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0, \quad \forall j=1, \dots, 2^m \quad (14)$$

$$\begin{bmatrix} P_{111} & P_{112} & 0 \\ P_{121} & P_{122} & \underline{\alpha}_i K_{(i)}^T \\ 0 & \underline{\alpha}_i K_{(i)} & \gamma \bar{u}_i^2 \end{bmatrix} \geq 0, \quad \forall i=1, \dots, m \quad (15)$$

$$\underline{\alpha}_i \in (0, 1], \quad \forall i=1, \dots, m \quad (16)$$

where  $K_{(i)}$  is the  $i$ -th row of  $K$ , and where,

$$\Omega_{1,j} = \begin{bmatrix} \Psi_{111} & \Psi_{112,j} & \Psi_{211} & \Psi_{212} & h_{\max}(W_{111}^T + P_{111}^T) & h_{\max}(W_{121}^T + P_{121}^T) \\ & \Psi_{122,j} & \Psi_{221,j} & \Psi_{222,j} & h_{\max}(W_{112}^T + P_{112}^T) & h_{\max}(W_{122}^T + P_{122}^T) \\ & & \Psi_{311} & \Psi_{312} & h_{\max}W_{211}^T & h_{\max}W_{221}^T \\ & & & \Psi_{322} & h_{\max}W_{212}^T & h_{\max}W_{222}^T \\ & & & & -h_{\max}R_{111} & -h_{\max}R_{112} \\ & & & & & -h_{\max}R_{122} \end{bmatrix} \quad (17)$$

$$\begin{bmatrix} h_{\max}(W_{311}^T + P_{211}^T) & h_{\max}(W_{321}^T + P_{221}^T) & -W_{311}^T A_h - \varepsilon_3 E_h^T E_h \\ h_{\max}(W_{312}^T + P_{212}^T) & h_{\max}(W_{322}^T + P_{222}^T) & -W_{312}^T A_h \\ h_{\max}(W_{411}^T + P_{311}^T) & h_{\max}(W_{421}^T + P_{321}^T) & -W_{411}^T A_h \\ h_{\max}(W_{412}^T + P_{312}^T) & h_{\max}(W_{422}^T + P_{322}^T) & -W_{412}^T A_h \\ -h_{\max}R_{211} & -h_{\max}R_{212} & 0 \\ -h_{\max}R_{221} & -h_{\max}R_{222} & 0 \\ -h_{\max}R_{311} & -h_{\max}R_{312} & 0 \\ -h_{\max}R_{321} & -h_{\max}R_{322} & 0 \\ & & -(\beta - 1)S + \varepsilon_3 E_h^T E_h \end{bmatrix} \quad (17)$$

$$\Omega_2 = \begin{bmatrix} P_{211}^T H_A + P_{221}^T L H_C & P_{211}^T H_h & W_{311}^T H_h \\ P_{212}^T H_A + P_{222}^T L H_C & P_{212}^T H_h & W_{312}^T H_h \\ P_{311}^T H_A + P_{321}^T L H_C & P_{311}^T H_h & W_{411}^T H_h \\ P_{312}^T H_A + P_{322}^T L H_C & P_{312}^T H_h & W_{412}^T H_h \end{bmatrix} \quad (18)$$

$$\Omega_3 = \text{diag}\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I\} \quad (19)$$

$$\Psi_{111} = (A + A_h)^T P_{211} + P_{211}^T (A + A_h) + (LC)^T P_{221} + P_{221}^T LC + W_{311}^T A_h + A_h^T W_{311} + S + \varepsilon_1 E_A^T E_A + (\varepsilon_2 + \varepsilon_3) E_h^T E_h$$

$$\Psi_{112,j} = (A + A_h)^T P_{212} + (LC)^T P_{222} + P_{211}^T B \Gamma_j(\underline{\alpha}) K + P_{221}^T (A - LC + B \Gamma_j(\underline{\alpha}) K) + A_h^T W_{312} + \varepsilon_1 E_A^T E_B \Gamma_j(\underline{\alpha}) K$$

$$\Psi_{122,j} = (B \Gamma_j(\underline{\alpha}) K)^T P_{212} + P_{212}^T B \Gamma_j(\underline{\alpha}) K + (A - LC + B \Gamma_j(\underline{\alpha}) K)^T P_{222} + P_{222}^T (A - LC + B \Gamma_j(\underline{\alpha}) K)$$

$$\begin{aligned}
& + \varepsilon_1 (E_B \Gamma_j(\underline{\alpha}) K)^T E_B \Gamma_j(\underline{\alpha}) K \\
\Psi_{211} &= P_{111}^T - P_{211}^T + (A + A_h)^T P_{311} + (LC)^T P_{321} + A_h^T W_{411}, \\
\Psi_{212} &= P_{121}^T - P_{221}^T + (A + A_h)^T P_{312} + (LC)^T P_{322} + A_h^T W_{412} \\
\Psi_{221,j} &= P_{112}^T - P_{212}^T + (B \Gamma_j(\underline{\alpha}) K)^T P_{311} + (A - LC + B \Gamma_j(\underline{\alpha}) K)^T P_{321} \\
\Psi_{222,j} &= P_{122}^T - P_{222}^T + (B \Gamma_j(\underline{\alpha}) K)^T P_{312} + (A - LC + B \Gamma_j(\underline{\alpha}) K)^T P_{322} \\
\Psi_{311} &= -P_{311} - P_{311}^T + h_{\max} \hat{A}_{h11,\max}^T R_{311} \hat{A}_{h11,\max}, \\
\Psi_{312} &= -P_{312} - P_{321}^T, \quad \Psi_{322} = -P_{322} - P_{322}^T
\end{aligned}$$

Under these conditions, system (10) is locally asymptotically stable for any initial condition  $\phi(\sigma)$  in the

$$\text{ball } \Phi(\sigma) = \left\{ \phi \in C_{h_{\max},n}^w; \|\phi\|_c^2 \leq \sigma \right\}, \text{ with } \sigma = \frac{1}{\gamma \pi_2} \quad (20)$$

$$\begin{aligned}
\text{where, } \pi_2 &= \lambda_{\max} \left( \begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix} \right) + \frac{h_{\max}}{(1-\beta)} \lambda_{\max}(S) \\
&+ \frac{3}{2} h_{\max}^2 \lambda_{\max} \left( \hat{A}_{h11,\max}^T R_{311} \hat{A}_{h11,\max} \right) \quad (21)
\end{aligned}$$

#### 4. Numerical example.

Consider system (1) with the following state space matrices:

$$\begin{aligned}
A &= \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H_A = \frac{1}{20} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
E_A &= [1 \ 1], \quad E_B = [1 \ 1], \quad \bar{u}_1 = 5, \quad \bar{u}_1 = 2 \quad (22)
\end{aligned}$$

with all the other matrices being equal to zero. It is exactly the same example that was used previously in [5], where the authors using relaxation techniques achieved an initial state set (assuming zero initial conditions for  $\hat{x}$ ),

$$D_0 = \left\{ x : x^T Z x \leq 1, Z = 10^{-6} \begin{bmatrix} 290.5 & -7.966 \\ -7.966 & 0.500 \end{bmatrix} \right\}.$$

The corresponding volume of this set was computed as  $\log(\text{Vol}(D_0)) = \log(\sqrt{\det(Z)}) = 11.61$ . In an attempt to achieve a larger set of initial conditions guaranteeing asymptotic stability of the closed-loop system, the following relaxation schemes were used in the present paper:

LMIR 1: Given  $K$ ,  $L$ ,  $\underline{\alpha}_1$  and  $\underline{\alpha}_2$ , solve for all the  $P$  matrices and  $\gamma$ , the problem  $\text{Min}(w \text{Trace}(P_i) + \gamma)$ , such that the LMIs of Theorem 1 are satisfied, where  $w$  is a column vector whose four entries are weights multiplying each diagonal entry of matrix  $P_i$  to form its trace.

LMIR 2: Given all the  $P$  matrices,  $K$ ,  $\underline{\alpha}_1$  and  $\underline{\alpha}_2$ , solve for  $L$  and  $\gamma$  the problem  $\text{Min}(\gamma)$ , such that the LMIs of Theorem 1 are satisfied.

Using the above relaxation schemes, the achieved set of initial conditions (assuming zero initial conditions for  $\hat{x}$ ),

$$\text{is } S_c = \left\{ x : x^T \gamma P_{111} x \leq 1, \gamma P_{111} = 10^{-6} \begin{bmatrix} 183.3 & -4.396 \\ -4.396 & 0.355 \end{bmatrix} \right\},$$

and the corresponding computed volume is  $\log(\text{Vol}(S_c)) = \log(\sqrt{\det(\gamma P_{111})}) = 11.90$ .

Figure 1 shows the sets of initial conditions achieved in [5] and in the present paper, where it is seen that there is a substantial increase in the size of the set of initial conditions guaranteeing asymptotic stability.

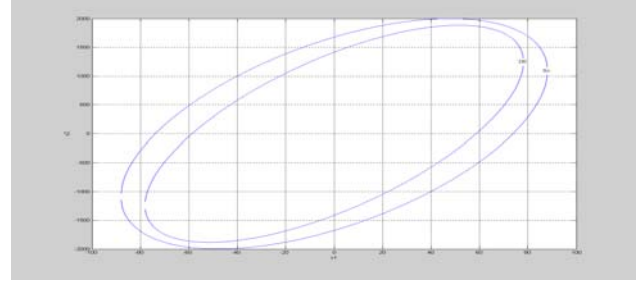


figure 1: Set  $S_c$  of initial conditions achieved in this paper as compared to set  $D_0$  of [5]

#### 5. Conclusion

The problem formulation employed here is believed to be the first and the most general considered so far with reference to the delay-dependent robust output feedback stabilization of state-delayed systems with saturating actuators. A major innovation of the approach adopted here is that the stabilizing control design is made dependent on both the value of the time-delay as well as on its rate of change. As it was demonstrated by way of an example, the result of Theorem 1 ensures stability for larger sets of initial conditions than previous results obtained in the literature.

#### References

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