An Improved Sufficient Condition for Robust ℓ_∞-Stability of Systems with Repeated Perturbations

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***Abstract*— This paper presents a new sufficient condition for robust ℓ_∞-stability of discrete-time systems with structured, repeated, linear time-varying, and induced ℓ_∞-norm bounded perturbations. The novel sufficient stability condition, which can be interpreted in terms of an augmented scaled small gain theorem, is shown to be less conservative than previous techniques.**

I. INTRODUCTION

The last two decades proliferated in ℓ_∞-stability (and performance) results; see [5] for a comprehensive summary of results and references. Recently, the class of robust ℓ_∞-stability problems involving *repeated*, linear time-varying (LTV), induced ℓ_∞-norm bounded perturbations have received particular attention, see [2], [3], and [4].

In the present paper, a novel sufficient stability condition for this type of problems is presented. The new sufficient stability condition can be interpreted in terms of an augmented scaled small gain theorem. The result is conceptually simple, yet essential as it permits a significant decrease in the conservativeness of design and analysis in many robustness problems (as compared with previous techniques).

It is also shown that the remarkable benefits of this new sufficient stability condition are unfortunately not transferable to its ℓ_2-stability counterpart.

The paper is outlined as follows. The required notation is introduced in §II. Then, the improved sufficient stability condition is presented in §III, followed by a thorough study of a justifying example problem in §IV. Finally, it is demonstrated in §V that the present result provides no advantages in the ℓ_2-stability case.

II. NOTATION

Let $\mathbb{Z}^+$ and $\mathbb{Z}^*$ denote the sets of positive and nonnegative integers, respectively.

Let $0_{m\times n}$ and I_n denote the zero matrix of dimension $m\times n$ and the identity matrix of dimension $n\times n$, respectively.

For any matrix $A \in \mathbb{R}^{m\times n}$, $A \triangleq [A_{ij}]_{\substack{i\in\{1,\ldots,m\}\\ j\in\{1,\ldots,n\}}}$, where A_{ij} is the ij^{th} entry of A. Similarly, for any vector $a \in \mathbb{R}^n$, $a \triangleq [a_1 \ldots a_n]^T$, where a_i is the i^{th} entry of a. This notation carries to the case of MIMO systems and vector signals.

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Let A^T and A^H denote the transpose and the conjugate transpose of the matrix A, respectively.

Let $\rho(A)$ denote the spectral radius of the square matrix A.

Let ℓ_p^n denote the space of all infinite sequences $\{s(k)\}_{k=0}^{\infty}$ of vectors of length n, $s(k) \in \mathbb{R}^n$, equipped with the norm

$$\|s\|_p \triangleq \sqrt[p]{\sum_{k=0}^{\infty}\sum_{i=0}^{n}|s_i(k)|^p} < \infty.$$

For $p = \infty$, also define $\|s\|_\infty \triangleq \sup_{k\geq 0}\max_{i\in\{1,\ldots,n\}}|s_i(k)|$. Given a bounded operator $S : \ell_p^n \mapsto \ell_p^m$ with $s \mapsto S(s)$, let

$$\|S\|_{p-ind} \triangleq \sup_{s\neq 0}\frac{\|S(s)\|_p}{\|s\|_p}$$

be the induced p-norm of S. Furthermore, if S is linear and causal, then $S(s)$ is determined by the convolution $(S * s)(k) \triangleq \sum_{l=0}^{k} S(k,l)s(l)$, where $S(k,l)$ denotes the kernel of S. In the case when S is also time-invariant, $S(s)$ simplifies to $(S*s)(k) \triangleq \sum_{l=0}^{k} S(l)s(k-l)$, where $\{S(l)\}_{l=0}^{\infty}$ is the impulse response of S. Then, it is known that, see [5], $\|S\|_{\infty-ind} = \|S\|_1$, where

$$\|S\|_1 \triangleq \max_{i\in\{1,\ldots,q\}}\sum_{j=1}^{p}\sum_{k=0}^{\infty}|S_{ij}(k)|. \qquad (1)$$

Moreover, let $\hat{S}(j\theta) \triangleq \sum_{k=0}^{\infty} S(k)e^{-j\theta k}$ denote the Fourier transform of the impulse response of S. Then, it is known that, see [5], $\|S\|_{2-ind} = \|\hat{S}\|_\infty$, where

$$\|\hat{S}\|_\infty \triangleq \sup_{\theta\in[0,2\pi]}\sqrt{\rho(\hat{S}(j\theta)\hat{S}^H(j\theta))}. \qquad (2)$$

III. AN IMPROVED SUFFICIENT CONDITION

For simplicity, only the following class of perturbations is considered throughout this paper. Given $n_\Delta \in \mathbb{Z}^+$,

$$\mathbf{\Delta}^{one} \triangleq \{\Delta = \delta I_{n_\Delta} : \delta \text{ is SISO, causal, and LTV}\}. \qquad (3)$$

Nevertheless, the results presented below extend in a straightforward manner to other classes of structured perturbations involving more than one repetition pattern.

Theorem 3.1: Let M be a square, discrete, causal, LTI system of size $n_M \in \mathbb{Z}^+$ and, given $n_a \in \mathbb{Z}^+$, define the augmented system $M_a \triangleq \begin{bmatrix} M & 0_{n_M\times n_a} \\ 0_{n_a\times n_M} & 0_{n_a\times n_a}\end{bmatrix}$. Let

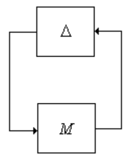

Fig. 1. $M\Delta$-loop

$n_\Delta = n_M$ and, for a given $\gamma \in \mathbb{R}$, let $\Delta \in \mathbf{\Delta}^{one}$ be such that $\|\Delta\|_{\infty-ind} < \frac{1}{\gamma}$. Assume that M and Δ are connected as in Fig.1. Then, if

$$\exists\ n_a \geq 0 \text{ and } D \in \mathbf{D}(n_a) \text{ s.t. } \|D^{-1}M_aD\|_1 \leq \gamma, \quad (4)$$

where $\mathbf{D}(n_a) \stackrel{\triangle}{=} \{[d_{ij}]_{\substack{i=\{1,\ldots,(n_M+n_a)\}\\ j=\{1,\ldots,(n_M+n_a)\}}} : d_{ij} \in \mathbb{R}, d_{11} = 1, 1 \geq d_{12} \geq \ldots \geq d_{1(n_M+n_a)} \geq 0\}$, then the $M\Delta$-loop is robust ℓ_∞-stable.

Proof: Follows directly from the scaled small gain theorem, see Appendix I for details. ■

Condition (4) can be best verified by the applications of global optimization techniques that are necessary because, for any given value of n_a,

$$\inf\{\|D^{-1}M_aD\|_1 : D \in \mathbf{D}(n_a)\}$$

is nonconvex and nondifferentiable with respect to the entries of D; see [4] and the references therein for further details.

Condition (4) will be referred to as the *standard sufficient condition* or the *augmented sufficient condition* when $n_a = 0$ or $n_a \geq 1$, respectively. Note that robust stability conditions similar to the standard sufficient condition are widely employed in the control literature, while the augmented sufficient condition has never been proposed or studied before.

It is easy to see that the augmented sufficient condition implies the standard sufficient condition for any given $n_a \geq 1$. Yet, an interesting and unexpected feature is that, as indicated by examples below, the augmented sufficient condition becomes less conservative as n_a increases.

IV. A Justifying Example Problem

A. Example Problem Statement

In order to demonstrate, beyond any doubt, the advantage of the augmented sufficient condition over its standard counterpart, an example problem is introduced as follows.

Let the system M' be characterized by the finite impulse response

$$\{M'(k)\}_{k=0}^{1} = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

By virtue of Theorem 3.1, the objective is to demonstrate that

$$\begin{aligned} &\exists\ n_a \geq 1 \text{ and } D \in \mathbf{D}(n_a) \text{ such that} \\ &\|D^{-1}M'_aD\|_1 < \inf_{D_0 \in \mathbf{D}(0)} \|D_0^{-1}M'D_0\|_1. \end{aligned} \quad (5)$$

To abridge the notation, define

$$\mathrm{E} \stackrel{\triangle}{=} \inf_{D_0 \in \mathbf{D}(0)} \|D_0^{-1}M'D_0\|_1.$$

The system M' was found to be one of the simplest examples which allow for the satisfaction of (5). Still, the resulting optimization problem E remains strongly nonconvex with respect to the entries of D_0. As a result, the strategy adopted to demonstrate (5) resides in the application of a complex global optimization algorithm (such as the branch-and-bound algorithm (BBA)) to derive an arbitrarily tight lower bound, $\underline{E}$, for E. Next, an integer $n_a \geq 1$ and a matrix $D \in \mathbf{D}(n_a)$ are sought such that

$$\|D^{-1}M'_aD\|_1 < \underline{E}.$$

Despite its significant complexity, the use of the BBA is fully justified as there is no simpler optimization technique that permits to derive the lower bound $\underline{E}$.

B. A Bounded Feasible Set

Proposition 4.1: For the example considered,

$$\mathrm{E} = \min_{D_0 \in \mathbf{D_b}(0)} \|D_0^{-1}M'D_0\|_1.$$

where $\mathbf{D_b} \stackrel{\triangle}{=} \{[d_{ij}]_{\substack{i=\{1,2\}\\ j=\{1,2\}}} : d_{ij} \in \mathbb{R}, d_{11} = 1, 1 \geq d_{12} \geq 0, |d_{21}| \leq 4, |d_{22}| \leq 4\}$ is a bounded subset of the original unbounded feasible set $\mathbf{D}(0)$.

Proof: Follows directly from Corollary 4.3 and Corollary 4.5 (see below). ■

The proposition above is important as it is now possible to apply a branch-and-bound algorithm over $\mathbf{D_b}$ in order to compute the global minimum for E, within any desired tolerance.

Proposition 4.2: Consider a set $\mathbf{D_1} \stackrel{\triangle}{=} \{[d_{ij}]_{\substack{i=\{1,2\}\\ j=\{1,2\}}} : d_{ij} \in \mathbb{R}, d_{11} = 1, 1 \geq d_{12} \geq 0, |d_{21}| \leq |d_{22}|\}$. Then,

$$\inf_{D \in \mathbf{D_1}} \|D^{-1}M'D\|_1 \geq \frac{|d_{22}^2 - 1|}{|d_{22}|}.$$

Corollary 4.3: If $|d_{22}| \geq 4$, then

$$\inf_{D \in \mathbf{D_1}} \|D^{-1}M'D\|_1 \geq 3.75.$$

Proof: (of Proposition 4.2)

From the definition of the scaling matrix,

$$\{D^{-1}M(k)D\}_{k=0}^{1} = \left\{ \frac{S(0)}{\det(D)}, \frac{S(1)}{\det(D)} \right\}, \quad (6)$$

where

$$S_{11}(0) \triangleq d_{11}d_{12} + d_{11}d_{22} - d_{12}d_{21} + d_{21}d_{22},$$
$$S_{12}(0) \triangleq d_{12}d_{12} + d_{22}d_{22},$$
$$S_{21}(0) \triangleq -d_{11}d_{11} - d_{21}d_{21},$$
$$S_{22}(0) \triangleq -d_{11}d_{12} + d_{11}d_{22} - d_{12}d_{21} - d_{21}d_{22},$$
$$S_{11}(1) \triangleq -d_{11}d_{12} - d_{11}d_{22} + d_{12}d_{21} + d_{21}d_{22},$$
$$S_{12}(1) \triangleq -d_{12}d_{12} + d_{22}d_{22},$$
$$S_{21}(1) \triangleq d_{11}d_{11} - d_{21}d_{21},$$
$$S_{22}(1) \triangleq d_{11}d_{12} - d_{11}d_{22} + d_{12}d_{21} - d_{21}d_{22}.$$

Let $D \in \mathbf{D_1}$. Then,

$$\|D^{-1}MD\|_1 \geq \frac{|d_{12}^2 + d_{22}^2| + |d_{12}^2 - d_{22}^2|}{|d_{11}d_{22} - d_{12}d_{21}|} \quad (7)$$
$$\geq \frac{|d_{12}^2 + d_{22}^2| + |d_{12}^2 - d_{22}^2|}{2|d_{22}|} \quad (8)$$
$$\geq \frac{|d_{22}^2 - 1|}{|d_{22}|}. \quad (9)$$

Equations (7), (8), and (9) follow respectively from: the definition of the ℓ_1 norm as applied to (6); the maximization of det(D) over $\mathbf{D_1}$; the second triangle inequality together with the definition of $\mathbf{D_1}$. ■

Proposition 4.4: Consider a set $\mathbf{D_2} \triangleq \{[d_{ij}]_{\substack{i=\{1,2\}\\ j=\{1,2\}}} : d_{ij} \in \mathbb{R}, d_{11} = 1, 1 \geq d_{12} \geq 0, |d_{21}| \geq |d_{22}|\}$. Then,

$$\inf_{D \in \mathbf{D_2}} \|D^{-1}M'D\|_1 \geq \frac{|d_{21}^2 - 1|}{|d_{21}|}.$$

Proof: Similar to the proof of Proposition 4.2. ■

Corollary 4.5: If $|d_{21}| \geq 4$, then

$$\inf_{D \in \mathbf{D_2}} \|D^{-1}M'D\|_1 \geq 3.75.$$

C. Computation of the Global Minimum of E *using a Branch-and-Bound Algorithm*

In this subsection, a BBA is developed to solve the problem

$$\mathrm{E}^* \triangleq \inf_{D \in \mathbf{D_b}} \|D^{-1}M'D\|_1.$$

The details of the customized BBA are given in Appendix III.

The numerical results are displayed in Fig.2. It can be seen that 54 iterations are required before the difference between the best upper bound and the worst lower bound reaches the desired tolerance (which is set to 10^{-4}). The final value of the worst lower bound of E^*, together with Proposition 4.1, guarantee that the global minimum of E is no less than 3.4142. Indeed, E yields a cost of $\sqrt{2}+2$ at $D_0 = \begin{bmatrix} 1 & \sqrt{2}-1 \\ -1 & \sqrt{2}-1 \end{bmatrix}$. For additional details concerning the complexity of E, see Appendix II.

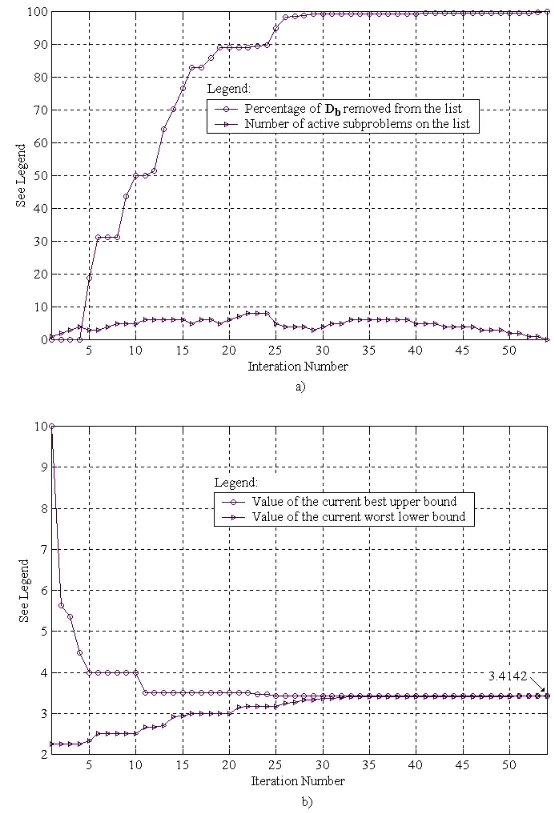

Fig. 2. Results using the branch-and-bound algorithm

D. Solution of the Augmented Example Problem

To complete this example, it remains to find an $n_a \geq 1$ and a $D \in \mathbf{D}(n_a)$ which satisfy (5). Given $n_a = 1$, one such solution is provided by the scaling matrix $D = \begin{bmatrix} 1.0000 & 0.4677 & 0.3041 \\ -0.6656 & 0.4677 & -0.7353 \\ -1.2546 & 0.7267 & 1.6175 \end{bmatrix}$ and yields a cost of 3.3100.

Extensive simulations demonstrate that the existence of a solution with a cost smaller than 3.3100 seems very unlikely even if values of n_a larger than one are considered.

V. An Augmented Robust ℓ_2-Stability Problem

As opposed to the robust ℓ_∞-stability problem formulation (4), consider the robust ℓ_2-stability problem defined by

$$\inf_{D \in \tilde{\mathbf{D}}(n_a)} \|D^{-1}\hat{M}_a D\|_\infty \leq \gamma, \quad (10)$$

where $\tilde{\mathbf{D}}(n_a) \triangleq \{D \in \mathbb{R}^{(n_M+n_a)\times(n_M+n_a)} : D = D^T, D > 0\}$ and let the other variables be defined as in §III; also see [11] for an introduction on robust ℓ_2-stability theory.

As it will be shown below, the remarkable benefits of the problem augmentation approach unfortunately do not carry over to the ℓ_2-stability case. Define the matrix

$$\tilde{D} \triangleq DD^T,$$

which is partitioned as follows

$$\tilde{D} \triangleq \begin{bmatrix} \tilde{D}_{(11)} & \tilde{D}_{(12)} \\ \tilde{D}_{(21)} & \tilde{D}_{(22)} \end{bmatrix},$$

where $D_{(11)} \in \mathbb{R}^{n_M \times n_M}$, $D_{(12)} \in \mathbb{R}^{n_M \times n_a}$, $D_{(21)} \in \mathbb{R}^{n_a \times n_M}$, and $D_{(22)} \in \mathbb{R}^{n_a \times n_a}$. Additionally, let

$$\tilde{D}_{(11)} \triangleq \bar{D}\bar{D}^T$$

and let

$$\begin{aligned}
A(j\theta) &\triangleq D^{-1}\hat{M}_a(j\theta)DD^T\hat{M}_a^H(j\theta)D^{-1^T}, \\
B(j\theta) &\triangleq DD^T\gamma^2 - \hat{M}_a(j\theta)DD^T\hat{M}_a^H(j\theta), \\
C(j\theta) &\triangleq \begin{bmatrix} \tilde{D}_{(11)} & \tilde{D}_{(12)} \\ \tilde{D}_{(21)} & \tilde{D}_{(22)} \end{bmatrix}\gamma^2 - \begin{bmatrix} \hat{M}(j\theta)\tilde{D}_{(11)}\hat{M}^H(j\theta) & 0 \\ 0 & 0 \end{bmatrix}, \\
D(j\theta) &\triangleq \tilde{D}_{(11)}\gamma^2 - \hat{M}(j\theta)\tilde{D}_{(11)}\hat{M}^H(j\theta).
\end{aligned}$$

Next, observe that

$$(10) \iff \inf_{D \in \tilde{\mathbf{D}}(n_a)} \sup_{\theta \in [0,2\pi]} \sqrt{\rho(A(j\theta))} \le \gamma \tag{11}$$

$$\iff \inf_{D \in \tilde{\mathbf{D}}(n_a)} \text{ s.t. } B(j\theta) \ge 0 \ \forall \theta \in [0, 2\pi] \tag{12}$$

$$\iff \inf_{\tilde{D} \in \tilde{\mathbf{D}}(n_a)} \text{ s.t. } C(j\theta) \ge 0 \ \forall \theta \in [0, 2\pi] \tag{13}$$

$$\implies \inf_{\tilde{D}_{(11)} \in \tilde{\mathbf{D}}(0)} \text{ s.t. } D(j\theta) \ge 0 \ \forall \theta \in [0, 2\pi] \tag{14}$$

$$\iff \inf_{\bar{D} \in \tilde{\mathbf{D}}(0)} \|\bar{D}^{-1}\hat{M}\bar{D}\|_\infty \le \gamma. \tag{15}$$

Equations (11), (12), (13), (14), and (15) follow respectively from: definition (2); the definition of $\tilde{D}$; LMI representation; the Schur complement Lemma; the definition of $\tilde{D}_{11}$, the notion of positive-definiteness, and again definition (2).

Thus, (10) implies that $\inf_{\bar{D} \in \tilde{\mathbf{D}}(0)} \|\bar{D}^{-1}\hat{M}\bar{D}\|_\infty \le \gamma$ so there is no advantage in considering $n_a \neq 0$ in the robust ℓ_2-stability case.

VI. Conclusion

An augmented sufficient condition for robust ℓ_∞-stability of systems with *repeated*, linear time-varying (LTV), induced ℓ_∞-norm bounded perturbation is presented in this paper.

It should be pointed out that the example problem investigated has been chosen for its simplicity rather than for its ability to allow the augmented sufficient condition to generate an impressive decrease in conservativeness as n_a increases. Still, in many cases, the improvement which imparts to the application of the augmented sufficient condition is substantial (as compared to the standard sufficient condition), see Appendix IV for such an example.

Interestingly, the motivation behind the development of the augmented sufficient condition originates from the study of robust ℓ_∞-stability of systems involving repeated perturbations under the perspective of the topological separation theory, see [8]. This connection will thus be investigated further and will hopefully provide new insight into the optimal choice of n_a for any given problem.

Appendix I
Proof of Theorem 3.1

Given $n_\Delta = n_M + n_a$, let the augmented perturbation block $\Delta_a \in \mathbf{\Delta}^{one}$ satisfy $\|\Delta_a\|_{\infty-ind} < \frac{1}{\gamma}$. Assume that M_a and Δ_a are connected in a similar fashion as M and Δ in Fig.1. Then, from the scaled small gain theorem, see [5], a sufficient condition for robust ℓ_∞-stability of the $M_a\Delta_a$-loop is given by (4). Observe that for any $\Delta_a \in \mathbf{\Delta}^{one}$ and any $D \in \mathbf{D}(n_a)$, the commutativity equation $\Delta_a D = D\Delta_a$ is also satisfied.

Moreover, since M_a and Δ_a are both block diagonal, the robust ℓ_∞-stability of the $M_a\Delta_a$-loop implies the robust ℓ_∞-stability of the $M\Delta$-loop. Hence, (4) is a valid sufficient condition for the robust ℓ_∞-stability of the $M\Delta$-loop. ■

Appendix II
Additional Information Concerning Problem E

It is interesting to note that E exhibits a local and a global minimum as well as a pair of saddle points. Both saddle points of E yield a cost of 4 and are located at $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The global minimum cost of E equals $\sqrt{2} + 2$ and is located at $D_0 = \begin{bmatrix} 1 & \sqrt{2}-1 \\ -1 & \sqrt{2}-1 \end{bmatrix}$ (Indeed, observe that $\{D_0{}^{-1}M(k)D_0\}_{k=0}^1 = \left\{\begin{bmatrix} 1 & \sqrt{2}-1 \\ -(\sqrt{2}+1) & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ which leads to $\|D_0{}^{-1}MD_0\|_1 = \sqrt{2}+2$). The local minimum of E yields a cost of ≈ 3.4641 at $D_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1.7321 \end{bmatrix}$. The saddle points and the local minimum are located on the boundary of the closed feasible set $\mathbf{D}(0)$.

For completeness, note that the corresponding necessary robust ℓ_∞-stability condition, see [2], yields a value of 2.9712 which is 13.0% less than $\sqrt{2} + 2$ (the global minimum of E).

Appendix III
Details of the Branch-and-Bound Algorithm

Since E* is nonconvex, it necessitates the use of a global optimization technique such as the BBA which is found to be particularly suitable for this case. The application of the BBA is never a simple task as the algorithm heavily relies on the development of a lower bound relaxed problem formulation. The last procedure will be explained in the context of E*.

A convenient notation will be found helpful. To this end, define

$$d \triangleq [d_{12}\ d_{d1}\ d_{22}]^T \in \mathbb{R}^3.$$

Also, let

$$\underline{d} \triangleq [\underline{d}_{12}\ \underline{d}_{d1}\ \underline{d}_{22}]^T \in \mathbb{R}^3$$

and

$$\overline{d} \triangleq [\overline{d}_{12}\ \overline{d}_{d1}\ \overline{d}_{22}]^T \in \mathbb{R}^3$$

be given lower and upper bounds for d, respectively. Corresponding to any pair $\underline{d}$ and $\overline{d}$, a subregion of $\mathbf{D_b}$ is defined by

$$\mathbf{D_b}(\underline{d},\overline{d}) \triangleq \mathbf{D_b} \cup \left\{ \left[\begin{array}{cc} 1 & d_{12} \\ d_{21} & d_{22} \end{array} \right] : \underline{d} \leq d \leq \overline{d} \right\}.$$

It is assumed that, at any stage of the BBA, the collection of subregions defined in terms of different selections of pairs $(\underline{d},\overline{d})$ provides a non-overlapping partition of $\mathbf{D_b}$.

With reference to each of the subregions $\mathbf{D_b}(\underline{d},\overline{d})$, the BBA proceeds to find an upper and a lower bound for the following subproblem

$$\mathrm{E}^*(\underline{d},\overline{d}) \triangleq \min_{D\in\mathbf{D_b}(\underline{d},\overline{d})} \|D^{-1}MD\|_1. \tag{16}$$

An upper bound for $\mathrm{E}^*(\underline{d},\overline{d})$ is very easy to obtain, in fact any $D \in \mathbf{D_b}(\underline{d},\overline{d})$ yields a valid upper bound for $\mathrm{E}^*(\underline{d},\overline{d})$. The real challenge is hence to find a lower bound for $\mathrm{E}^*(\underline{d},\overline{d})$ which is sought in terms of a solution to a relaxed subproblem $\underline{\mathrm{E}}^*(\underline{d},\overline{d})$ such that

$$\underline{\mathrm{E}}^*(\underline{d},\overline{d}) \leq \mathrm{E}^*(\underline{d},\overline{d})$$

and

$$\lim_{\|\underline{d}-\overline{d}\|\to 0} \underline{\mathrm{E}}^*(\underline{d},\overline{d}) = \lim_{\|\underline{d}-\overline{d}\|\to 0} \mathrm{E}^*(\underline{d},\overline{d}) \quad \forall(\underline{d},\overline{d}). \tag{17}$$

Ideally, the relaxed subproblem $\underline{\mathrm{E}}^*(\underline{d},\overline{d})$ needs to be defined in such a way as to allow for a rapid global solution. Typically, $\underline{\mathrm{E}}^*(\underline{d},\overline{d})$ is required to be at least strictly-quasiconvex.

The main idea underlying the BBA, see [10], is to refine the partitioning of $\mathbf{D_b}$ in a way permitting to find a subregion which contains the global solution of E^* by comparing all the upper and lower bounds for $\mathrm{E}^*(\underline{d},\overline{d})$ corresponding to all subregions $\mathbf{D_b}(\underline{d},\overline{d})$. This process of successive partitioning of $\mathbf{D_b}$ is carried out as follows. A list of non-overlapping subregions $\mathbf{D_b}(\underline{d},\overline{d})$ is first created and the corresponding subproblems $\mathrm{E}^*(\underline{d},\overline{d})$ are solved (approximately) for each member on the list in terms of their lower an upper bounds. For any given subregion $\mathbf{D_b}(\underline{d},\overline{d})$ on the list, if the computed lower bound for $\mathrm{E}^*(\underline{d},\overline{d})$ exceeds any upper bound for any other subregion on the list, then $\mathbf{D_b}(\underline{d},\overline{d})$ can be removed from the list of subregions as it cannot contain the optimal solution over $\mathbf{D_b}$. On the other hand, if $\mathbf{D_b}(\underline{d},\overline{d})$ remains on the list, then further partitioning of $\mathbf{D_b}(\underline{d},\overline{d})$ is carried out enriching the pool of subregions on the list. This process is continued until all the subregions on the list exhibit lower bounds which differ from the currently best available upper bound by the required tolerance margin for the solution error. Naturally, throughout the optimization process, the global solution E^* is always guaranteed to belong to the interval whose limits are the current best available upper bound and the current worst available lower bound among all subproblems on the list.

It stands out that the relaxed subproblem $\underline{\mathrm{E}}^*(\underline{d},\overline{d})$ is, by far, the most complicated component of the BBA. Hence, a computable expression for $\underline{\mathrm{E}}^*(\underline{d},\overline{d})$ is derived below preceded by some helpful lemmas.

Lemma 3.1: [9]

$i)$ Given $\underline{x},\underline{y},\overline{x},\overline{y} \in \mathbb{R}$, define the convex set

$$\begin{aligned} C(\underline{x},\underline{y},\overline{x},\overline{y}) \triangleq \{&(x,y,z) \in \mathbb{R}^3 : z \geq \overline{y}x + \overline{x}y - \overline{xy}, \\ &z \leq \underline{y}x + \overline{x}y - \overline{x}\underline{y}, z \leq \overline{y}x + \underline{x}y - \underline{x}\overline{y}, \\ &z \geq \underline{y}x + \underline{x}y - \underline{xy}\}. \end{aligned}$$

If $\underline{x} \geq x \geq \overline{x}$, $\underline{y} \geq y \geq \overline{y}$, and $z = xy$, then $(x,y,z) \in C(\underline{x},\underline{y},\overline{x},\overline{y})$.

$ii)$ Similarly, given $\underline{x},\overline{x} \in \mathbb{R}$, define the convex set

$$\begin{aligned} C(\underline{x},\overline{x}) \triangleq \{&(x,z) \in \mathbb{R}^2 : z \geq 2\overline{x}x - \overline{x}^2, \\ &z \leq \overline{x}x + \underline{x}x - \underline{x}\overline{x}, z \geq 2\underline{x}x - \underline{x}^2\}. \end{aligned}$$

If $\underline{x} \geq x \geq \overline{x}$ and $z = x^2$, then $(x,z) \in C(\underline{x},\overline{x})$.

Lemma 3.2: [1]

Let $X \subset \mathbb{R}^n$ be a convex set and let the function $g_1 : X \mapsto \mathbb{R}$ be linear and the function $g_2 : X \mapsto \mathbb{R}$ be linear and positive on X. Then, the function $g : X \mapsto \mathbb{R}$, defined by $g(x) \triangleq \frac{g_1(x)}{g_2(x)}$ is strictly-quasiconvex on X.

Lemma 3.3: [1]

Let $X \subset \mathbb{R}^n$ be a convex set and, let $g_i : X \mapsto \mathbb{R}$, $i \in \{1,...,m\}$, $m \in \mathbb{Z}^+$, be a family of strictly-quasiconvex functions on X. Then, the function $g : X \mapsto \mathbb{R}$, defined by $g(x) \triangleq \max_{i\in\{1,...,m\}} g_i(x)$ is also strictly-quasiconvex on X.

Given $\underline{d},\overline{d} \in \mathbb{R}^3$, (16) is redefined as follows to stress the dependance on the optimization variables

$$f(d) \triangleq \mathrm{E}^*(\underline{d},\overline{d}). \tag{18}$$

The above problem is further rearranged by replacing the bilinear terms of (18) by new variables defined as

$$d_r \triangleq [d_{12;12}\ d_{12;21}\ d_{21;21}\ d_{21;22}\ d_{22;22}]^T \in \mathbb{R}^5$$

and satisfying

$$\begin{aligned} d_{12;12} &\triangleq d_{12}d_{12}, \\ d_{12;21} &\triangleq d_{12}d_{21}, \\ d_{21;21} &\triangleq d_{21}d_{21}, \\ d_{21;22} &\triangleq d_{21}d_{22}, \\ d_{22;22} &\triangleq d_{22}d_{22}. \end{aligned} \tag{19}$$

Thus, the subproblem (18) becomes

$$g(d,d_r) \text{ subject to (19)}$$

where, assuming that (19) holds, $g(d,d_r) = f(d)$.

Given the following convex set of constraints derived from Lemma 3.1

$$\begin{aligned}
&(d_{12}, d_{12;12}) \in C(\underline{d}_{12}, \overline{d}_{12}),\\
&(d_{21}, d_{21;21}) \in C(\underline{d}_{21}, \overline{d}_{21}),\\
&(d_{22}, d_{22;22}) \in C(\underline{d}_{22}, \overline{d}_{22}),\\
&(d_{12}, d_{21}, d_{12;21}) \in C(\underline{d}_{12}, \underline{d}_{21}, \overline{d}_{12}, \overline{d}_{21}),\\
&(d_{21}, d_{22}, d_{21;22}) \in C(\underline{d}_{21}, \underline{d}_{22}, \overline{d}_{21}, \overline{d}_{22}),
\end{aligned} \tag{20}$$

a relaxed version for the subproblem is given by

$$g(d, d_r) \text{ subject to (20)}$$

where again, assuming that (20) holds, $g(d, d_r) \leq f(d)$.

Partitionning the relaxed feasible set according to the sign of $d_{22} - d_{12;21}$, the final expression for the relaxed subproblem is given by

$$\underline{\mathrm{E}}(\underline{d}, \overline{d}) \triangleq \min\left(h_1(d, d_r), h_2(d, d_r)\right)$$

where

$$h_1(d, d_r) \triangleq g(d, d_r) \text{ subject to (20) and } d_{22} - d_{12;21} \geq 0$$

and

$$h_2(d, d_r) \triangleq g(d, d_r) \text{ subject to (20) and } d_{22} - d_{12;21} \leq 0.$$

It follows from Lemma 3.2 and 3.3 that $h_1(d, d_r)$ and $h_2(d, d_r)$ are both strictly quasiconvex optimization problems. Hence, the relaxed subproblem $\underline{\mathrm{E}}^*(\underline{d}, \overline{d})$ is easily solvable (two steps required) by employing any nondifferentiable optimization algorithm (Shor's r-algorithm has been used here). Moreover, from Lemma 3.1, it is easy to show that $\underline{\mathrm{E}}^*(\underline{d}, \overline{d})$ satisfies condition (17).

APPENDIX IV
ADDITIONNAL EXAMPLE

Consider the discrete, causal, stable, LTI system M of dimension compatible with $\Delta \in \mathbf{\Delta}^{one}$ such that $\|\Delta\|_{\infty-ind} < \frac{1}{\gamma}$, as illustrated in Fig.1. The system M is fully characterized by the following finite impulse response:

$$\{M(k)\}_{k=0}^{3} \triangleq \left\{ \begin{bmatrix} 2.3 & 3.4 \\ -1.9 & 0.7 \end{bmatrix}, \begin{bmatrix} -1.3 & 0.5 \\ 2.0 & -0.6 \end{bmatrix}, \begin{bmatrix} 1.9 & 2.9 \\ 1.2 & 4.6 \end{bmatrix}, \begin{bmatrix} 0.2 & -3.3 \\ 3.8 & 4.8 \end{bmatrix} \right\}.$$

Stability of the $M\Delta$-loop is assessed using Theorem 3.1.

Four different n_a values ($n_a = 0, 1, 2, 3$) are considered in this example. The n_a-dependent problems of the form

$$\inf\{\|D^{-1} M_a D\|_1 : D \in \mathbf{D}(n_a)\} \tag{21}$$

are solved by using a local nonsmooth optimization algorithm. For each value of n_a, a hundred local searches are performed over (21). Each local search is initialized by a randomly chosen scaling matrix D (with its entries limited to the interval $[-5, 5]$).

TABLE I
OPTIMAL RESULTS

Value of n_a	Minimal Cost
$n_a = 0$	16.35
$n_a = 1$	15.84
$n_a = 2$	15.76
$n_a = 3$	15.76

The results are displayed in Table I, where it is shown that the minimum cost of 15.76 is obtained with $n_a = 2$ and $n_a = 3$. It is an improvement of 3.61% as compared with 16.35 which is the minimal cost computed when $n_a = 0$ (i.e., the smallest value associated with the standard sufficient condition). The importance of these results is yet better elucidated by comparing the costs listed in Table I with 13.30 which is the value derived from the necessary robust ℓ_∞-stability condition computed for this problem according to the methodology proposed in [2]. It then follows that the proposed augmented sufficient condition is able to tighten the gap with its necessary counterpart by $19.36\% = \left(1 - \frac{15.76-13.30}{16.35-13.30}\right) \cdot 100\%$.

For the purpose of further comparisons, the next values have also been calculated: $\|M\|_1 = 19.60$ and $\rho\begin{pmatrix} \|M_{11}\|_1 & \|M_{12}\|_1 \\ \|M_{21}\|_1 & \|M_{22}\|_1 \end{pmatrix} = 18.01$. The two previous expressions respectively follows from sufficient conditions for robust ℓ_∞-stability of systems with unstructured perturbations and independent (i.e., not repeated) structured perturbations; see [6] and [7] for details.

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