Computational Aspects of a Criterion for Robust ℓ_∞-Stability of Systems with Repeated Perturbations

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***Abstract*— The class of robust stability problems considered involves structured, *repeated*, linear time-varying, induced ℓ_∞-norm bounded perturbations. A well known sufficient condition for these robust problems follows from the scaled small gain theorem and is stated in the form of a scaled ℓ_1-norm minimization problem with block diagonal scaling matrices. A procedure reducing the domain in the search for an optimal scaling matrix, while preserving global optimality, is presented here.**

I. Introduction

Various classes of robust ℓ_∞-stability (and performance) problems are addressed in the control literature, see [4] for a comprehensive summary of results and references. Amongst those, [5], [6], and [7] consider structured, independent, linear time-varying (LTV) and/or nonlinear, induced ℓ_∞-norm bounded perturbations. The purpose of this paper is to generalize the previous contributions to include the case where the perturbations are possibly *repeated*. Problems with repeated perturbations are frequent in applications as they naturally arise, for example, in applications comprising several identical components, see [1], [2], [3].

It is well known that sufficient conditions for robust ℓ_p-stability problems follow from the scaled small gain theorem and can be cast in the form of scaled induced ℓ_p-norm minimization problems. In presence of repeated perturbations, the scaling matrices are block diagonal. In the ℓ_2-stability case, only Hermitian positive-definite scaling matrices need to be considered, see [8]. A similar conclusion is not applicable to robust ℓ_∞-stability problems. Hence, the main contribution of this paper is a result which justifies a significant reduction of the optimization domain (i.e., the set of admissible scaling matrices) for the ℓ_1 (i.e., induced ℓ_∞) problem involved, while preserving global optimality. Such domain reduction is applicable to both the pure robust stability analysis and the robust controller synthesis problems. This result is especially useful for efficient implementation of global optimization algorithms which often need to be used to solve these types of problems, see [3] and [7]. The reduction of the optimization domain is shown to be substantial and results in a proportional decrease in computational effort needed to achieve global optimality.

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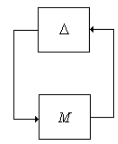

Fig. 1. $M\Delta$-loop

II. Notation and Problem Statement

Let $\mathbb{Z}^+$ and $\mathbb{Z}^*$ denote the sets of positive and nonnegative integers, respectively.

Let I_n denote the identity matrix of dimension $n \times n$.

Given a set $\mathbf{A}$, let $\mathrm{card}\{\mathbf{A}\}$ denote its cardinality.

Let ℓ_∞^n denote the space of all infinite sequences $\{s(k)\}_{k=0}^\infty$ of vectors of length n, $s(k) \in \mathbb{R}^n$, equipped with the norm $\|s\|_\infty \triangleq \sup_{k\geq 0} \max_{i\in\{1,\ldots,n\}} |s_i(k)|$. Given a bounded operator $S : \ell_\infty^n \mapsto \ell_\infty^m$ with $s \mapsto S(s)$, let $\|S\|_{\infty-\mathrm{ind}} \triangleq \sup_{s\neq 0} \frac{\|S(s)\|_\infty}{\|s\|_\infty}$ be the induced ∞-norm of S. In the case when S is linear, causal, and time-invariant, then it is known that, see [4], $\|S\|_{\infty-\mathrm{ind}} = \|S\|_1$, where

$$\|S\|_1 \triangleq \max_{i\in\{1,\ldots,m\}} \sum_{j=1}^{n} \sum_{k=0}^{\infty} |S_{ij}(k)|.$$

Given $n \in \mathbb{Z}^+$ and $p_I \in \mathbb{Z}^+$, $I \in \{1,...,n\}$, define the following classes of perturbations and scaling matrices:

$$\mathbf{\Delta} \triangleq \{\mathrm{diag}(\delta_1 I_{p_1}, ..., \delta_n I_{p_n}) : I \in \{1,...,n\}\}, \quad (1)$$

where δ_i, $I \in \{1,...,n\}$, is a discrete, causal, SISO, LTV, perturbation sub-block, and

$$\mathbf{D} \triangleq \{\mathrm{diag}(D^1, ..., D^n) : D^I \in \mathbb{R}^{p_I\times p_I}, I \in \{1,...,n\}\}, \quad (2)$$

where $D^I \triangleq [D^I_{ij}]_{\substack{i\in\{1,\ldots,p_I\} \\ j\in\{1,\ldots,p_I\}}}$.

Let M be a discrete, causal, LTI system of dimension compatible with $\Delta \in \mathbf{\Delta}$, $\|\Delta\|_{\infty-ind} < 1$ as illustrated by Fig.1. By virtue of the scaled small gain theorem: if there exists a $D \in \mathbf{D}$ such that $\|D^{-1}MD\|_1 \leq 1$, then the $M\Delta$-loop is robust ℓ_∞-stable. Such satisficing problem is often solved in terms of the more strict global optimization problem:

$$GOP \triangleq \inf_{D\in\mathbf{D}} \|D^{-1}MD\|_1.$$

In general, the *GOP* is nonconvex and therefore necessitates the use of global optimization techniques. The associated computational effort is proportional to the size of the set of scaling matrices **D** (i.e., the optimization domain).

III. Reduction of the Scaling Set

The following theorem provides a method for significantly reducing the cardinality of $\mathbf{D}$ without compromising global optimality of the *GOP*.

Theorem 3.1: There exists a proper subset $\tilde{\mathbf{D}} \subset \mathbf{D}$ (independent of M) of cardinality $\operatorname{card}\{\tilde{\mathbf{D}}\} = \operatorname{card}\{\mathbf{D}\}\Big/\prod_{I=1}^{n}(2^{p_I}p_I!)$ such that

$$(GOP \triangleq) \inf_{D\in\mathbf{D}} \|D^{-1}MD\|_1 = \inf_{\tilde{D}\in\tilde{\mathbf{D}}} \|\tilde{D}^{-1}M\tilde{D}\|_1. \quad (3)$$

Such a subset can be sought in the form

$$\tilde{\mathbf{D}} \triangleq \{\text{diag}(D^1,...,D^n), D^I \in \mathbb{R}^{p_I\times p_I}, \quad (4)$$
$$D^I_{11} \geq D^I_{12} \geq ... \geq D^I_{1p_I} \geq 0, I \in \{1,...,n\}\}.$$

Proof: For brevity of exposition, the proof is limited to full block scaling matrices (i.e., for a given fixed $p_I \in \mathbb{Z}^+$, $\mathbf{D} \triangleq \{D : D \in \mathbb{R}^{p_I\times p_I}\}$ with $M \in \ell_1^{p_I\times p_I}$).

The theorem is proved by constructing a $\tilde{\mathbf{D}}$ with the required properties. Define the following two sets of matrices

$$\mathbf{D_{CS}} \triangleq \Big\{D^p \in \mathbb{R}^{p_I\times p_I} : D^p_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \neq p \\ -1, & i = j = p \end{cases},$$
$$p \in \{1,...,p_I\}\Big\},$$

$$\mathbf{D_{CP}} \triangleq \Big\{D^{pq} \in \mathbb{R}^{p_I\times p_I} : p,q \in \{1,...,p_I\}, p \neq q,$$
$$D^{pq}_{ij} = \begin{cases} 1, & ij = pq \vee ij = qp \\ 0, & ij = pp \vee ij = qq \\ 1, & i = j \wedge (ij \neq pp \vee ij \neq qq) \\ 0, & i \neq j \wedge (ij \neq pq \vee ij \neq qp) \end{cases}\Big\}.$$

Let $D^p \in \mathbf{D_{CS}}$ and $D^{pq} \in \mathbf{D_{CP}}$. Note that the matrix transfer function MD^p is equal to a modification of M obtained by multiplying by -1 the p^{th} column of M and MD^{pq} is equal to another modification of M obtained by permuting the p^{th} and the q^{th} columns of M. Furthermore,

$$\|(D^p)^{-1}MD^p\|_1 = \|(D^{pq})^{-1}MD^{pq}\|_1 = \|M\|_1. \quad (5)$$

Define $\bar{\mathbf{D}}_{\mathbf{CS}} \triangleq \{D_1\cdots D_i\cdots D_n \in \mathbb{R}^{p_I\times p_I} : D_i \in \mathbf{D_{CS}},\ \forall n < \infty\}$; i.e., $\bar{\mathbf{D}}_{\mathbf{CS}}$ contains all finite products with, possibly repeated, elements of $\mathbf{D_{CS}}$. Similarly, define $\bar{\mathbf{D}}_{\mathbf{CP}} \triangleq \{D_1\cdots D_i\cdots D_n \in \mathbb{R}^{p_I\times p_I} : D_i \in \mathbf{D_{CP}},\ \forall n < \infty\}$. Observe that $\bar{\mathbf{D}}_{\mathbf{CS}}(\supset \mathbf{D_{CS}})$ and $\bar{\mathbf{D}}_{\mathbf{CP}}(\supset \mathbf{D_{CP}})$ are finite sets with respective cardinalities of 2^{p_I} and $p_I!$. Also, the set

$$\bar{\mathbf{D}} \triangleq \{D_1\cdots D_i\cdots D_n \in \mathbb{R}^{p_I\times p_I} : D_i \in \bar{\mathbf{D}}_{\mathbf{CS}} \cup \bar{\mathbf{D}}_{\mathbf{CP}},$$
$$\forall n < \infty\}, \quad (6)$$

where $\bar{\mathbf{D}}(\supset \bar{\mathbf{D}}_{\mathbf{CS}} \cup \bar{\mathbf{D}}_{\mathbf{CP}})$ is a finite set of self-invertible matrices and with cardinality $2^{p_I}p_I!$. Additionally, from (5), any $\bar{D} \in \bar{\mathbf{D}}$ satisfies $\|\bar{D}^{-1}M\bar{D}\|_1 = \|M\|_1$. Hence, for any given scaling matrix $D' \in \mathbb{R}^{p_I\times p_I}$, there exists $2^{p_I}p_I! - 1$ scaling matrices $D = D'\bar{D}$, $\bar{D} \in \bar{\mathbf{D}}\backslash I_{p_I}$, such that

$$\|D^{-1}MD\|_1 = \|\bar{D}D'^{-1}MD'\bar{D}\|_1 = \|D'^{-1}MD'\|_1. \quad (7)$$

This suggests the existence of a minimal subset of scaling matrices where only one out of $2^{p_I}p_I!$ scaling matrices needs to be considered. Such a subset is constructed below.

Let $D \in \mathbf{D}$ be a given scaling matrix. Corresponding to D, there always exists a $\bar{D} \in \bar{\mathbf{D}}_{\mathbf{CS}}$ such that $\tilde{D} = D\bar{D}$ with all $\tilde{D}_{1i} \geq 0$, $i \in \{1,...,p_I\}$ (obtained by adequate column sign changes). Additionally, there always exists a $\bar{\bar{D}} \in \bar{\mathbf{D}}_{\mathbf{CP}}$ such that $\tilde{D} = D\bar{D}\bar{\bar{D}}$ with $\tilde{D}_{11} \geq \tilde{D}_{12} \geq ... \geq \tilde{D}_{1p_I} \geq 0$ (obtained by suitable column permutations). Furthermore, if $D \in \mathbf{D}$ is such that $\|D^{-1}MD\|_1 = \epsilon$, for some $\epsilon \geq 0$, then for the $\bar{D}$ and $\bar{\bar{D}}$ just constructed, $\|\bar{\bar{D}}\bar{D}D'^{-1}MD'\bar{D}\bar{\bar{D}}\|_1 = \epsilon$, by virtue of (7).

The above correspondence between each $D \in \mathbf{D}$ and $\tilde{D}$ defines a non-invertible mapping $F : \mathbf{D} \mapsto \tilde{\mathbf{D}}$, where $\tilde{D} \triangleq F(D)$. It follows by construction that $\tilde{\mathbf{D}} \triangleq \{\tilde{D} \in \mathbb{R}^{p_I\times p_I} : \tilde{D}_{11} \geq \tilde{D}_{12} \geq ... \geq \tilde{D}_{1p_I} \geq 0\}$. It is also an easy exercise to verify that $\tilde{\mathbf{D}}$ has cardinality $\operatorname{card}\{\mathbf{D}\}/2^{p_I}p_I!$. Additionally, by construction, for each $D \in \mathbf{D}$ there exists a $\tilde{D} \in \tilde{\mathbf{D}}$ such that $\|D^{-1}MD\|_1 = \|\tilde{D}^{-1}M\tilde{D}\|_1$, so (3) follows automatically. ■

The subset $\tilde{\mathbf{D}}$ is non-unique. However, the proposed $\tilde{\mathbf{D}}$ of the form (4) has the advantage of being a convex set which contains the identity scaling matrix ($I \in \tilde{\mathbf{D}}$). Additionally, such a $\tilde{\mathbf{D}}$ includes as a subset $\tilde{\mathbf{d}} \triangleq \{\operatorname{diag}(\tilde{d}^1,...,\tilde{d}^n) : \tilde{d}^I \in \mathbb{R}^+, I \in \{1,...,n\}\}$ which is encountered in [6] and [7] for the special case when $\mathbf{\Delta} \triangleq \{\operatorname{diag}(\delta_1,...,\delta_n) : I \in \{1,...,n\}\}$, i.e., for the case when there are no repeated perturbations in $\mathbf{\Delta}$.

IV. Conclusion

The proposed domain reduction procedure, stated in terms of Theorem 3.1, allows for a significant decrease in the computational effort required to carry out the global optimization for the associated GOP. See [1] and [3] for illustrative examples.

It is an easy exercice to modify Theorem 3.1 as to widen its applicability to the class of robust ℓ_∞-stability problem involving structured perturbation blocks defined as in (1), but with MIMO δ_i sub-blocks.

References

[1] Cadotte, Michalska, Boulet,“Design of a Robust Controller for a Heat Exchanger System with ℓ_∞-Performance Objectives and Repeated Perturbations”, *WSEAS Transaction on Systems*, Issue 5, Vol. 3, 2004.

[2] Cadotte, Michalska, Boulet,“A Necessary Condition for Robust ℓ_∞-Stability of Systems with Repeated Perturbations”, *CDC* 2004.

[3] Cadotte, Michalska, Boulet,“An Improved Sufficient Condition for Robust ℓ_∞-Stability of Systems with Repeated Perturbations”, *ACC* 2005.

[4] Dahleh, Diaz-Bobillo, *Control of Uncertain Systems: a Linear Programming Approach*, Prentice Hall, 1995.

[5] Khammash, Pearson, “Performance Robustness of Discrete-Time Systems with Structured Uncertainty”, *IEEE TAC*, vol. 36, no. 4, 1991.

[6] Khammash, Pearson, “Analysis and Design for Robust Performance with Structured Uncertainty”, *Syst & Contr Lett*, 20, 1993.

[7] Khammash, Salapaka, Van Voorhis, “Robust synthesis in ℓ_1: a globally optimal solution”, *IEEE TAC*, vol. 46, no. 11, 2001.

[8] Zhou, Doyle, Glover, *Robust and optimal control*, Prentice-Hall, Upper Saddle River, 1996.