Quantized and Sampled Control of Linear Second Order Systems

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Abstract — Continuous systems are today controlled digitally. It is therefore necessary to consider the effects of quantization and sampling. We show that any second-order LTI system can be controlled exactly by fixed quantized feedback independently from the resolution of the sensors. When sampling is considered, only practical stabilization can be achieved and the size of the limit cycle depends on the sampling rate.

I. INTRODUCTION

Sampling and quantization effects have a special importance in control engineering since, today, analog control is employed only exceptionally. Since the very beginnings of digital control technology, research was concerned with the effects of sampling [1], [2], and quantization [3], [4]. Traditionally, the effects of quantization were approached by considered quantization errors as disturbances, for example, characterized as uniformly distributed noise [4]. The latter approach is advocated in contemporary texts on sampled-data systems [5].

Prior attempts to analyze the properties of quantized feedback systems include those of Delchamps [6], who found that systems with quantized feedback cannot be stabilized in the traditional sense. However, under the special assumption that the quantization step can be made arbitrarily small, Brockett and Liberzon showed that asymptotic stability can still be achieved [7]. When variable-rate sampling instants can be made to coincide in time with quantization, other convergence results were shown by Kofman [8]. Logarithmic quantization was considered by Elia and Mitter to show that the quantization step can be made arbitrarily small, but since $\varsigma$ represents a signed quantity, $\frac{1}{2}$ has to be added to ensure symmetry of the quantization error with respect to zero. Most analog-to-digital converters, “mid-rise” types, are designed to behave this way so that $\varsigma$ and $-\varsigma$ have the same magnitude after quantization. When the origin is arbitrary, as in the case of incremental motion encoders, the origin can still be unambiguously put in correspondence between the continuous domain, $\varsigma \in \mathbb{R}$, and the quantized domain, $\{\lfloor \varsigma \rfloor + \frac{1}{2} \} - \frac{1}{2} \in \mathbb{Z}$.

III. EFFECT OF QUANTIZATION

A. Equilibrium of a Quantized Feedback System

We must first define an appropriate notion of equilibrium for systems such as (2). The right-hand-side of (2), evaluated at $x = 0$, is equal to $-\frac{1}{2}(k_1 + k_2), k_1, k_2 > 0$. Hence, the origin is not a stationary point in an ordinary sense. In state space form, the motion of system (2) is governed by the discontinuous vector field $v(x) = [v_1(x), v_2(x)]^T \in \mathbb{R}^2$, $x = [x_1, x_2]^T \in \mathbb{R}^2$, where $x_1 \triangleq x$ and $x_2 \triangleq \dot{x}$.

$$v_1(x) = x_2,$$

$$v_2(x) = a_2x_2 + a_1x_1 - k_2([x_2] + \frac{1}{2}) - k_1([x_1] + \frac{1}{2}).$$ (3)

The points of discontinuity in (3) are all the points in the set $D \triangleq \{x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}\}$. For further analysis, it is convenient to partition the state space into disjoint “tiles” within which the system vector field $v$ is continuous,

$$S^{i,j} \triangleq \{x_1 \in [i, i+1], x_2 \in [j, j+1]\}, \quad i, j \in \mathbb{Z}.$$

The closures of these sets are

$$\bar{S}^{i,j} \triangleq \{x_1 \in [i, i+1], x_2 \in [j, j+1]\}, \quad i, j \in \mathbb{Z}.$$
It is convenient to introduce

\[ v^{i,j} : x \mapsto \left[ v^{i,j}_1(x), v^{i,j}_2(x) \right]^T, \]

\[ \tilde{v}^{i,j} : x \mapsto \left[ \tilde{v}^{i,j}_1(x), \tilde{v}^{i,j}_2(x) \right]^T, \]

where a \( v^{i,j} \) is the restriction of \( v \) to the tile \( S^{i,j} \), and \( \tilde{v}^{i,j} \) is the continuous extension of the \( v^{i,j} \) to \( S^{i,j} \).

An approach to modeling systems such as \( \dot{x} = v(x) \) is to consider them as governed by differential inclusions \([10]\), rather than by differential equations. These are written, \( \dot{x} \in F(x) \).

The set-valued mapping \( F : x \mapsto F(x) \subset \mathbb{R}^2, \forall x \in \mathbb{R}^2 \), is related to the piecewise continuous vector field \( v(x) \) as follows. When \( x \in S^{i,j} \) the RHS of (4) reduces to a singleton, \( F(x) = \{ v^{i,j}(x) \} \). For points on the boundaries of \( S^{i,j} \), \( x \in \partial S^{i,j} \), for some \( i, j \in \mathbb{Z} \), the set \( F(x) \) is determined as the closed convex-hull of all the limit points \( v^* \) of the vector field \( v \) on the domain \( S = \bigcup S^{i,j} \), where

\[ v^* = \lim_{\xi \to x, x \in S} \{ v(\xi) \}. \]  

A solution to (4) is defined as an absolutely continuous function \( x : t \mapsto x(t) \in \mathbb{R}^2 \) such that \( \dot{x}(t) \in F(x(t)) \) almost everywhere. Then, see \([10]\).

**Definition 1**: A point, \( x_E \), is a generalized stationary point of a quantized system such as (2) iff \( 0 \notin F(x_E) \).

With this definition, the constant trajectory \( x(t) = x_E \) satisfies the differential inclusion (4). In particular, the origin is a generalized stationary point of (2) since \( F(0) = \frac{1}{2} \co[[0, \varepsilon_1], [0, \varepsilon_2]]^T \cup \{ 0 \} \) is nonempty. For example, let \( r > 0 \). Then \( F(x) = \co \{ 0, \varepsilon_1, \varepsilon_2 \}^T \) with

\[ v^{n,0}_2 = (a_1 - k_1)n + a_1 r - \frac{1}{2}k_1 - \frac{1}{2}k_2, \]

\[ v^{n,-1}_2 = (a_1 - k_1)n + a_1 r - \frac{1}{2}k_1 + \frac{1}{2}k_2. \]

We show that \( 0 \notin F([n, r, 0]^T) \) by demonstrating that \( v^{n,0}_2 \) and \( v^{n,-1}_2 \) always have the same sign.

From (6), \( a_1 - k_1 < 0 \). From (7)-(8), \( v^{n,0}_2 \) and \( v^{n,-1}_2 \) have the same signs \( \forall n \in \mathbb{N} \setminus \{ 0 \} \), provided that both quantities are negative when \( n = 0 \). In the latter case, again by (6),

\[ v^{0,0}_2 = a_1 r - \frac{1}{2}k_1 - \frac{1}{2}k_2 < 0, \]

\[ v^{0,-1}_2 = a_1 r - \frac{1}{2}k_1 + \frac{1}{2}k_2 < 0. \]

If now \( n = -1 \), then from (6),

\[ v^{1,0}_2 = - a_1 + k_1 + a_1 r - \frac{1}{2}k_1 - \frac{1}{2}k_2 > 0, \]

\[ v^{1,-1}_2 = - a_1 + k_1 + a_1 r - \frac{1}{2}k_1 + \frac{1}{2}k_2 > 0. \]

As \( a_1 - k_1 < 0 \), one can see that \( v^{n,0}_2, v^{n,-1}_2 \) are all negative from (7)-(8). On the other hand, if \( n \leq 1 \) these quantities are all positive. Hence, \( 0 \notin F([n, 0]^T) \), for all \( n \in \mathbb{Z} \setminus \{ 0 \} \).

**Corollary 1**: The sole constant trajectory of system (2) is \( x(t) = 0 \), for all \( t \).

**B. Stability of a 2nd-Order System with Quantized Feedback**

At this point it is noted that, provided that Condition (6) holds, the formalism of differential inclusions is no longer needed since, away from the origin no ambiguity arises as to how the system traverses the discontinuities of the defining vector field. From the proof of Proposition 1, it can be seen that the trajectories of the system form a correct phase portrait that encircles the origin in the clockwise direction.

Under Condition (6) further analysis can hence be conducted by employing a standard system representation as in (2) in the region \( R_D = \mathbb{R}^2 \setminus \{ 0 \} \). The main stability result is easier to derive by first demonstrating that the closed-loop quantized system is input-output passive.

**Proposition 2 (Passivity)**: A sufficient condition for the second-order system with quantized PD feedback (2) to be input-output passive with respect to the output \( y = \dot{x} \) and the dissipation rate \( \dot{x}u \) is that the gains satisfy

\[ k_2 > 4 \max \{ a_2, a_1 \} \]

(9)

**Proof**: On \( R_D \), consider the augmented system

\[ \ddot{x} = a_2 \dot{x} + a_1 x - k_2 (|\dot{x}| + \frac{1}{2}) - k_1 (|x| + \frac{1}{2}) + u, \]

and the associated input-output product

\[ \dot{x} u = \ddot{x} a_1 \dot{x} - a_2 \dot{x} + \dot{x} k_1 (|x| + \frac{1}{2}) + \dot{x} k_2 (|\dot{x}| + \frac{1}{2}). \]

Integrating (11) over an arbitrary interval \([t_1, t_2]\) yields

\[ \int_{t_1}^{t_2} \dot{x} u \, dt = \int_{t_1}^{t_2} \ddot{x} \, dt - a_1 \int_{t_1}^{t_2} \dot{x} \, dt - a_2 \int_{t_1}^{t_2} \dot{x} \, dt + k_2 \int_{t_1}^{t_2} \dot{x} (|\dot{x}| + \frac{1}{2}) \, dt \]

\[ + k_1 \int_{t_1}^{t_2} \dot{x} (|x| + \frac{1}{2}) \, dt + \dot{x} (|\dot{x}| + \frac{1}{2}) \, dt \]

\[ = \frac{1}{2} [\dot{x}^2 (t_2) - \dot{x}^2 (t_1)] - a_1 \frac{1}{2} [x^2 (t_2) - x^2 (t_1)] + \frac{1}{2} [\dot{x}^2 (t_2) + \dot{x}^2 (t_1)] \]

\[ + k_2 \int_{t_1}^{t_2} \dot{x} (|\dot{x}| + \frac{1}{2}) \, dt - a_2 \int_{t_1}^{t_2} \dot{x}^2 \, dt + k_2 \int_{t_1}^{t_2} \dot{x} (|\dot{x}| + \frac{1}{2}) \, dt. \]

It is now convenient to introduce the quantization error function, \( \xi \mapsto \Gamma(\xi) = \xi - |\xi| \). Then,

\[ \int_{t_1}^{t_2} \dot{x} \, dt + \frac{1}{2} \, dt = \int_{t_1}^{t_2} \dot{x} (x - \Gamma(x) + \frac{1}{2}) \, dt. \]
It is seen that
\[
\int_{t_1}^{t_2} \dot{x}(x(t)) \, dt = \int_{x(t_1)}^{x(t_2)} \Gamma'(\xi) \, d\xi = \int_{x(t_1)}^{x(t_2)} [x(t_2)]' - [x(t_1)]' - \Gamma'(x(t_1)) \, d\xi = \frac{1}{2} [x(t_2)]^2 + \Gamma^2(x(t_2)) - [x(t_1)]^2 - \Gamma^2(x(t_1)).
\] (12)

Hence,
\[
\int_{t_1}^{t_2} \dot{x}(\xi) + \frac{1}{2} \, d\tau = \frac{1}{2} x^2(t_2) - \frac{1}{2} x^2(t_1) + \frac{1}{2} [\Gamma(x(t_2)) - \Gamma^2(x(t_2))] - \frac{1}{2} [\Gamma(x(t_1)) - \Gamma^2(x(t_1))].
\]

Now we show that a positive lower bound exists for the term
\[
P(\dot{x}) \triangleq -a_2 \int_{t_1}^{t_2} \dot{x}^2 \, d\tau + k_2 \int_{t_1}^{t_2} (\dot{x}^2 + \frac{1}{2} k_2 \dot{x}) \, d\tau
\]
on the set \( R_D \). Without loss of generality, we may assume that \( t_1, t_2 \) are such that \( \forall t \in [t_1, t_2] \) either
\[
\begin{align*}
(a) & \quad \dot{x}(t) = +n + r(t), n \in \mathbb{N} + \{0\}, \text{ or} \\
(b) & \quad \dot{x}(t) = -n - r(t), n \in \mathbb{N}.
\end{align*}
\]

Case (a): The value of term \( P(\dot{x}) \), for \( r \in [0, 1] \), is
\[
P(\dot{x}) = \int_{t_1}^{t_2} \left[ -a_2 n + r(\tau) \right] \, d\tau + k_2 n \left( n + \frac{1}{2} \right) \, d\tau.
\]
If \( n = 0 \), then \( \dot{x}(\tau) = r(\tau) \in [0, 1] \) and
\[
P(\dot{x}) = \int_{t_1}^{t_2} (-a_2 \dot{x}^2 + \frac{1}{2} k_2 \dot{x}) \, d\tau.
\]
Choosing \( k_2 \geq 4 \max\{0, a_2\} \) and because \( \dot{x} \in [0, 1] \),
\[
P(\dot{x}) \geq \int_{t_1}^{t_2} (-a_2 \dot{x}^2 + \frac{1}{2} k_2 \dot{x}) \, d\tau \geq \frac{1}{4} k_2 \int_{t_1}^{t_2} \dot{x}^2 \, d\tau.
\]
Now, if \( n \in \mathbb{N} \),
\[
P(\dot{x}) = \int_{t_1}^{t_2} \left[ -a_2 n^2 + 2a_2 n r - a_2 r^2 + k_2 n + \frac{1}{2} k_2 r \right] \, d\tau
\]
\[
\geq \left[ -a_2 n^2 - a_2 n + k_2 n \right] \int_{t_1}^{t_2} \tau \, d\tau
\]
\[
= \left[ -a_2 n^2 - a_2 n + k_2 n \right] \left( t_2 - t_1 \right)
\]
\[
= \left[ -a_2 n^2 - a_2 n + k_2 n \right] \left( t_2 - t_1 \right)
\]
\[
\int_{t_1}^{t_2} \dot{x}^2 \, d\tau + \frac{1}{2} k_2 n \left( t_2 - t_1 \right)
\]
\[
\geq \left[ -a_2 n^2 - a_2 n + k_2 n \right] \left( t_2 - t_1 \right)
\]
\[
\frac{1}{4} k_2 \int_{t_1}^{t_2} \dot{x}^2 \, d\tau.
\]
When \( n \in \mathbb{N}, n \geq 2, r \in [0, 1] \),
\[
-a_2 (-n + r) + k_2 (-n + \frac{1}{2}) < 0, \Rightarrow (a_2 - k_2)n - a_2 r + \frac{1}{2} k_2 < 0.
\]
Provided that \( k_2 \geq 4 \max\{0, a_2\} \) then
\[
(a_2 - k_2)n - a_2 r + \frac{1}{2} k_2 \leq -\frac{1}{2} k_2 n.
\]
Finally, the bound for case (b) is
\[
P(\dot{x}) \geq \int_{t_1}^{t_2} \left[ -a_2 (-n + r) + k_2 (-n + \frac{1}{2}) \right] (-n + r) \, d\tau
\]
\[
\geq \frac{1}{4} k_2 n \left( n + 1 \right) \left( 2 t_2 - t_1 \right).
\] (13)

Combining (12)-(13) gives
\[
P(\dot{x}) \geq \left\{
\begin{array}{ll}
(\min \left\{ \frac{1}{2} k_2; - (a_2 - k_2) \right\} n^2 \left( t_2 - t_1 \right),
\end{array}
\right.
\]
\[
\int_{t_1}^{t_2} \dot{x} \, d\tau = \frac{1}{2} [x^2(t_2) - x^2(t_1)] - a_2 [x^2(t_2) - x^2(t_1)]
\]
\[
+ k_1 [x^2(t_2) - x^2(t_1)] + k_1 [\Gamma(x(t_2)) - \Gamma^2(x(t_2))]
\]
\[
= k_1 [\Gamma(x(t_1)) - \Gamma^2(x(t_1))]
\]
\[
- a_2 \int_{t_1}^{t_2} \dot{x}^2 \, d\tau + k_2 \int_{t_1}^{t_2} (\dot{x}^2 + \frac{1}{2} \dot{x}) \, d\tau
\]
\[
= \frac{1}{2} [x^2(t_2) - x^2(t_1)] - \frac{1}{2} (a_1 - k_2) [x^2(t_2) - x^2(t_1)]
\]
\[
+ \frac{1}{2} k_2 (\dot{x}(t_2) - \dot{x}(t_1))
\]
\[
- \frac{1}{2} k_1 [\Gamma(x(t_1)) - \Gamma^2(x(t_1))]
\]
\[
\left\{
\begin{array}{ll}
\int_{t_1}^{t_2} \dot{x} \, d\tau \geq \frac{1}{4} k_2 \int_{t_1}^{t_2} \dot{x}^2 \, d\tau.
\end{array}
\right.
\]
(14)

It follows that
\[
P(\dot{x}) \geq \min \left\{ \frac{1}{2} k_2; - (a_2 - k_2) \right\} n^2 \left( t_2 - t_1 \right),
\]
which proves input-output passivity of (2) under conditions (9) with the storage function
\[
S(x, \dot{x}) \triangleq \frac{1}{2} [x^2 + (k_1 - a_1)x^2 + k_1 (\Gamma(x) - \Gamma^2(x))].
\] (15)

This storage function is well defined as it is positive definite since, for all \( x, \Gamma(x) \in [0, 1] \) so \( \Gamma(x) - \Gamma^2(x) \geq 0 \). Also, \( S \) is locally Lipschitz continuous. The latter is established by showing Lipschitz continuity of the term \( \Gamma(\cdot) - \Gamma^2(\cdot) \) at points \( x \in \mathbb{Z} \). Let \( x = n \) for any \( n \in \mathbb{Z} \) and consider a neighbourhood \( |n - \epsilon, n + \epsilon|, \epsilon \in [0, \frac{1}{2}] \) together with two points \( n - \epsilon, n + \epsilon \in \mathbb{N}, \epsilon \in [0, 1] \). We note that
\[
[n - \epsilon, n + \epsilon] \supset [n - 1, n + 1] \supset \Gamma(n - 1, n + 1)
\]
\[
[n - \epsilon, n + \epsilon] \subset [n, n + \epsilon] \supset \Gamma(n + \epsilon, n + 1)
\]
i.e. \( \epsilon_1 = 1 - \Gamma(n - 1) \) and i.e. \( \epsilon_2 = \Gamma(n + \epsilon) \). Thus
\[
[\Gamma(n + \epsilon) - \Gamma^2(n + \epsilon) - \Gamma(n - 1, n + 1)]
\]
\[
\leq \Gamma(n + \epsilon) - \Gamma^2(n + \epsilon) + \Gamma(n - 1) - \Gamma^2(n - 1)
\]
\[
\leq (1 - \epsilon_2) \epsilon_2 (1 - \epsilon_1) \epsilon_1 \leq \epsilon_2 + \epsilon_1.
\]

Substituting back for \( x = n \) shows that for any \( \epsilon_1, \epsilon_2 < \epsilon \)
\[
[\Gamma(x + \epsilon_2) - \Gamma^2(x + \epsilon_2) - \Gamma(x - \epsilon_1) + \Gamma^2(x - \epsilon_1)]
\]
\[
\leq [(x + \epsilon_2) - (x - \epsilon_1)],
\]
demonstrating Lipschitz continuity of \( \Gamma(\cdot) - \Gamma^2(\cdot) \) with a Lipschitz constant equal to one.
Passivity entails asymptotic stability of the closed loop quantized system as shown below.

**Theorem 1 (Stability):** A second-order system with quantized PD feedback (2) is globally asymptotically stable if the gains satisfy Conditions (9).

**Proof:** Setting $u = 0$ in (15) and adopting the storage function (16) shows that

$$0 = S(\dot{x}(t_2), x(t_2)) - S(\dot{x}(t_1), x(t_1)) + P(\dot{x}),$$

so that on $R_T$,

$$S(\dot{x}(t_2), x(t_2)) - S(\dot{x}(t_1), x(t_1)) \leq -P(\dot{x}) < 0. \quad (17)$$

Since the storage function is proper, i.e., $S(x, \dot{x}) \rightarrow \infty$ as $||x, \dot{x}|| \rightarrow \infty$, the level sets of $S$ are compact and are invariant because $S$ is decreasing monotonically along the trajectories of (2). Hence, the system is stable.

Since $t_1, t_2$ are arbitrary, it follows from (17) and the properties of the storage function, along with the lower bound in (14) and continuity of $\dot{x}$, that the trajectories of (2) approach the largest invariant set contained in the set

$$Z \triangleq \{(x, \dot{x}) \in \mathbb{R}^2 | P(\dot{x}) = 0\} = \{(x, \dot{x}) \in \mathbb{R}^2 | \dot{x} = 0\}.$$

We will show that the largest invariant set contained in $Z$ is the constant, zero trajectory. Suppose that $x(t) \neq 0$ for some $t \in [t_1, t_2]$, but that $y(t) = \dot{x}(t) \equiv 0$ for all $t \in [t_1, t_2]$. Then, $\dot{x}(t) \equiv 0$ for all $t \in [t_1, t_2]$ which implies that $x(t) = x(t) \neq 0$ for all $t \in [t_1, t_2]$, i.e., $\dot{x}(t)$ must be a stationary point of (2). This contradicts the result of Proposition 1 which states that the only stationary point is $x(t) = 0$.

Hence, $Z = \{0\}$ and every trajectory of (2) approaches the origin. This proves asymptotic stability of the system. ■

**IV. ILLUSTRATIVE EXAMPLE**

The stabilization result of Theorem 1 is somewhat counterintuitive as one could think that a quantized feedback can only provide for practical stabilization (to a one-quantum neighborhood of the origin). As its stands, the result establishes asymptotic stabilization regardless of the size of the quantum. A simple example illustrates this phenomenon.

Consider the quantized PD control of a double integrator, i.e., assume that $a_1 = a_2 = 0$ in (2). Let the initial condition be $[x_1(0), x_2(0)] = [0, c]$ where $c > 0$ is such that $0, c \in \{x_1, x_2\} \subset S(x_1, x_2) < 1$, (less than one quantum away from 0). Referring to Fig. 1, quadrant by quadrant, the solution of (2) is a concatenation of portions of trajectories,

1. $\dot{x}_1 = x_2, \dot{x}_2 = -\frac{1}{2}k_1 - \frac{1}{2}k_2,$
2. $\dot{x}_1 = x_2, \dot{x}_2 = \frac{1}{2}k_1 + \frac{1}{2}k_2,$
3. $\dot{x}_1 = x_2, \dot{x}_2 = \frac{1}{2}k_1 + \frac{1}{2}k_2,$
4. $\dot{x}_1 = x_2, \dot{x}_2 = +\frac{1}{2}k_1 - \frac{1}{2}k_2.$

For simplicity, denote $a \triangleq \frac{1}{2}(k_1 + k_2)$, and $b \triangleq \frac{1}{2}(k_1 - k_2)$ and assume that $b < a$ as guaranteed by Condition (6) of Proposition 1. Note that all four subsystems are unstable. In each quadrant, the fragments of the system trajectory emanating from the initial condition $(0, c)$ are easily computed. The points $[x_1^*, 0], [0, x_2^*]$ are the points of “scheduled switches” corresponding to switching instants $t^*$. Also, $c < \sqrt{2}$ as implied by $S(0, c) < 1$, such that in all cases $S(x_1^*, 0) < 1$ and $S(0, x_2^*) < 1$, as expected.

V. THE EFFECT OF SAMPLING

Actual digital controllers measure the output at discrete instants of time. Given a sampling period, $T$, the system with quantized and sampled control (2) becomes, $x_k = x(kT), \dot{x}_k = a_2 \dot{x} + a_1 x - k_2(\lfloor x_k \rfloor + \frac{1}{2}) - k_1(\lfloor x_k \rfloor + \frac{1}{2})$, \quad (18)

$\Delta v_2(x, \dot{x}), \forall t \in [kT, (k + 1)T], \forall k \in \mathbb{N}$.

The stationary points of (18) are still derived according to Definition 1 where the set-valued mapping $F : x \mapsto F(x) \subset \mathbb{R}^2, \forall x \in \mathbb{R}^2$, is now calculated by taking limits as $x_k \rightarrow x$, i.e., the limit points $v^* \Delta v$ on the domain $S = \bigcup S^{n, 1}$, when $x \in \partial S^{n, 1}$, are now assumed to be computed as

$$v^* = \lim_{x \rightarrow x, x_k \in S} v(x_k) \quad (19)$$

![Fig. 1. Phase portrait of the double integrator with quantized PD controller.](image-url)
which replaces (5). With this modification the stationary points of the quantized and sampled system remain the same, that is, we can show what follows.

**Proposition 3:** A sufficient condition for the origin to be the sole stationary point of (18) is given by (6).

**Proof:** As in the proof of Proposition 1, for all $x$ such that $|x_2| > 0$, $0 \notin F(x)$, since any $v \in F(x)$ has the form $v = [x_2, 1]^\top \neq 0$. Adopting (19) to calculate the set-valued mapping $x \mapsto F(x)$, it can be verified that the vector fields $\hat{v}^{i,j} \in F(x); i = 0, -1, n, n - 1; j = 0, -1$ are the same as those found in the proof of Proposition 1. Hence, points of the form $[0, n + r]^\top, r \in [0, 1], n \in \mathbb{Z}$ and of the form $[0, n]^\top, n \in \mathbb{Z} - \{0\}$ are not stationary points of (18).

Although passivity of the system is lost due to sampling, and so is asymptotic stability, it is possible to show what constitutes a type of robustness result for the feedback controller which employs only quantization. The system (18) is still practically stable in the following sense.

**Theorem 2:** Suppose that the gains of the feedback controller in (18) satisfy, as before,

$$k_2 > 4 \max\{0, a_2\}, \quad k_1 > k_2 + 2 \max\{0, a_1\}.$$  

(20)

Then the quantized and sampled feedback control system (18) is practically stable in the sense that for every pair of level sets $S_\epsilon, S_\gamma$ with $\epsilon > 0$ of the storage function,

$$S_\epsilon \triangleq \{\{x_1, x_2\} \mid S(x_1, x_2) \leq \epsilon\},$$

there exists a sufficiently small sampling period $T$ such that all system trajectories of (18) originating in $S_\epsilon$ reach $S_\epsilon$ in finite time and remain in $S_\epsilon$ thereafter.

**Proof:** For any given $\epsilon > 0$, let $K_\gamma > 0$ be a bound for the RHS of (18) on the compact level set $S_\epsilon$, i.e.

$$|\sigma(x, \dot{x})| \leq K_\gamma,$$

and also, $|\dot{x}| \leq \sqrt{2\gamma}$.  

(21)

forall $x, \dot{x}$, $x_k, \dot{x}_k \in S_\gamma$, where, WLOG, $\gamma > 1, K_\gamma > 1$ and $\epsilon < \frac{\sqrt{2\gamma}}{4}$.

The sampled system is now

$$\dot{x}_a = a_2 \dot{x}_a + a_2 \dot{x}_a - k_1(x_a + \frac{1}{2}) + k_1([-x_a + \frac{1}{2}] + k_2([-x_a] + k_1([-x_a]),\dot{x}_a),\forall t \in [kT, (k + 1)T], \forall k \in \mathbb{N}.$$  

(22)

Along the solutions of this system the dissipation inequality no longer holds as, on $R_D$, it is replaced by

$$S(\dot{x}(t_2), x(t_2)) - S(\dot{x}(t_1), x(t_1)) \leq -P(\dot{x}) + k_2 \int_{t_1}^{t_2} |([x_a] - [x_a])| \dot{x} \, \mathrm{d}t + k_1 \int_{t_1}^{t_2} |([x_a] - [x_a])| \dot{x} \, \mathrm{d}t.$$  

(23)

Now, let $\{t_i\}_{i \in \mathbb{N}}$ be a sequence of instants $t_{i+1} = t_i + \Delta t, \Delta t = \frac{1}{4N_{\gamma}, \gamma}, < \frac{1}{4}$, for some $h$ to be selected later. From the Mean Value Theorem,

$$\dot{x}(t_2) - \dot{x}(t_1) = [\dot{x}(t_i + \theta)| \Delta t < h, \theta \in [0, \Delta t]$$  

(24)

$$|\dot{x}(t_2) - \dot{x}(t_1)| = |\dot{x}(t_i + \theta)| \Delta t < h, \forall i \in \mathbb{N},$$

whenever $[x(t_i), \dot{x}(t_i)]^\top \in S_\gamma$. Note that

$$[\dot{x}(t) - \dot{x}(t)] \neq 0 \lor [x(t) - x(t)] \neq 0,$$

only when $[x(t), \dot{x}(t)]^\top \in S_n^{n_1, n_2}$ and $[x(kT), \dot{x}(kT)]^\top \in S_n^{n_1, n_2}$ for different tiles of the state space. Now suppose that the trajectory of (22) satisfies $[x(t), \dot{x}(t)]^\top \in S_{C_\gamma} \cap S_\gamma$ for $\tau \in [t_i, t_{i+1}]$, where $C_{C_\gamma}$ denotes the complement of the level set $S_\gamma$. By virtue of (24) and since the origin is the sole stationary point of (22), there exists a constant $h$ determining $\Delta t$ such that $[x(t), \dot{x}(t)]$ change value at most once in the interval $[t_i, t_{i+1}]$ and such that $[x(t), \dot{x}(t)]^\top \in S_\gamma, \forall \tau \in [t_i, t_{i+1}]$ whenever $[x(t_i), \dot{x}(t_i)]^\top \in S_{C_\gamma}$. The value of $h$ decreases to zero as $\epsilon \to 0$.

For the value of $h$ selected as above, let $T = \frac{\pi}{\Delta t}$ which implies that sampling occurs at each time instant $t_i, i \in \mathbb{N}$, $m$ to be selected later. The positive terms on the RHS of (23) can be bounded as follows,

$$k_2 \int_{t_i}^{t_{i+1}} |([x_a] - [x_a])| \dot{x} \, \mathrm{d}t \leq k_2 \int_{t_i}^{t_{i+1}} |([x_a] - [x_a])| |\dot{x}| \, \mathrm{d}t \leq k_2 T \sqrt{2\gamma},$$  

(25)

for some $j \in [0, m - 1], j \in \mathbb{N}$.

Similarly, on $R_D$,

$$k_1 \int_{t_i}^{t_{i+1}} |([x_a] - [x_a])| \dot{x} \, \mathrm{d}t \leq k_1 T \sqrt{2\gamma}.$$  

We now need a lower bound for $P(\dot{x})$ in (23). From (14) and (20) it follows that

$$P(\dot{x}) \geq \begin{cases} \frac{1}{2} k_2 (2\sigma - 2\sigma_1), & \text{if } |\dot{x}(t)| \in [n, n + 1], n \in \mathbb{N}, \forall \tau \in \sigma_1, \sigma_2, \\ \frac{1}{2} k_2 \int_{\sigma_1}^{\sigma_2} \dot{x} \, \mathrm{d}t, & \text{if } |\dot{x}(t)| \in [0, 1], \forall \tau \in \sigma_1, \sigma_2. \end{cases}$$  

(26)

Using (24), it is easy to see that, if at $t_i, |\dot{x}(t_i)| \geq 1$ then $|\dot{x}(t)| \geq 1 - K_\gamma \Delta t > \frac{1}{4}$ for all $\tau \in [t_i, t_{i+1}]$ and

$$P(\dot{x}) \geq \frac{1}{2} k_2 (\sigma_1 - \sigma_2) + \frac{1}{2} k_2 \int_{\sigma_1}^{\theta} \dot{x} \, \mathrm{d}t \geq \frac{1}{2} k_2 \Delta t$$  

where $\sigma$ corresponds to the time instant at which $|\dot{x}(\tau)| = 1$.

Now, suppose that there exists a $\sigma \in [t_i, t_{i+1}]$ such that $|\dot{x}(\tau)| = 0$. Under the Conditions (20), it follows from the proof of Proposition 1 that there exist constants $\psi > 0$ and $\delta \in [0, 1]$ such that, whenever $|\dot{x}| \leq \delta$,

$$a_1 x + a_2 \dot{x} - k_1([x_a + \frac{1}{2}] + \frac{1}{2}) + \frac{1}{2} k_2 \leq -\psi x + 0,$$

$$a_1 x + a_2 \dot{x} - k_1([x_a + \frac{1}{2}] + \frac{1}{2}) + \frac{1}{2} k_2 \leq \psi x + 0.$$  

Also, it follows from (24) that for any $n \in \mathbb{N}$

$$|\dot{x}(t) - \dot{x}(kT)| \leq 2T K_\gamma, \tau \in (k - 1)T, (k + 1)T,$$

$$|x(t) - x(kT)| \leq 2 \sqrt{2\gamma} T, \tau \in (k - 1)T, (k + 1)T.$$  

whenever $[x(t), \dot{x}(t)]^\top, [x_k, \dot{x}_k]^\top \in S_\gamma$. Combining the above, and assuming that $T$ is sufficiently small or, equivalently, that $m$ is greater than some $m^*$, $\forall k \in \mathbb{N} : |\dot{x}_k| \leq \delta$.
and \( \forall \tau \in [(k-1)T, (k+1)T) \) such that \( |\dot{x}(\tau)| \leq \delta \)

\[
v_2(x(\tau), \dot{x}(\tau)) = a_1(x(\tau) + a_2\dot{x}(\tau) - k_1(|x(kT)| + \frac{1}{2}) + \frac{1}{2}k_2
\]

\[
= a_1(x(T) + a_2\dot{x}(T) - k_1(|x(kT)| + \frac{1}{2}) + \frac{1}{2}k_2 + a_1(x(\tau) - x(kT)) + a_2(x(\tau) - x(kT))
\]

\[
\leq -\psi + |a_1|2\sqrt{2mT} + |a_2|2TK\gamma \leq -\frac{1}{2}\psi \text{ if } x(\tau) > 0,
\]

\[
v_2(x(\tau), \dot{x}(\tau)) = a_1(x(\tau) + a_2\dot{x}(\tau) - k_1(|x(kT)| + \frac{1}{2}) - \frac{1}{2}k_2
\]

\[
= a_1(x(T) + a_2\dot{x}(T) - k_1(|x(kT)| + \frac{1}{2}) - \frac{1}{2}k_2 + a_1(x(\tau) - x(kT)) + a_2(x(\tau) - x(kT))
\]

\[
\geq -\psi - |a_1|2\sqrt{2mT} - |a_2|2TK\gamma \geq \frac{1}{2}\psi \text{ if } x(\tau) < 0.
\]

The above implies that the system trajectories traverses the set \( N_\delta = \{(x, \dot{x}) \in \mathbb{R}^2 | |\dot{x}| \leq \delta \} \) in finite time; i.e., \( N_\delta \) is not an invariant set of the system. From (21) the time \( t_R \) needed for the system trajectories to traverse \( N_\delta \) is estimated as \( t_R \geq \frac{2\psi}{\psi} \leq t_B \). WLOG and not prejudicing the previous choice of \( \gamma \), it can now be assumed that \( \Delta t = 2at \). If there exists a \( \sigma \in [kT, (k+1)T) \) such that \( \dot{x}(\sigma) = 0 \), then

\[
P(\dot{x}) \geq \frac{1}{2}k_2 \int_{t_k}^{t_{k+1}} x^2 \, dt \geq \frac{1}{2}k_2 \int_{t_k}^{t_{k+1}} (\delta - K_\gamma \tau)^2 \, dt,
\]

\[
\geq \frac{1}{2}k_2 \delta_0^2 \frac{1}{K_\gamma^2} = \frac{1}{12}k_2 \delta_0^2 \Delta t.
\]

By virtue of (25), (26), (27), there exists a \( c > 0 \) such that

\[
S(x(t_{k+1}), \dot{x}(t_{k+1})) - S(x(t_k), \dot{x}(t_k)) \leq -\frac{1}{12}k_2 \delta_0^2 \Delta t + \sqrt{2\gamma(k_1 + k_2) \Delta t} \leq -c\Delta t,
\]

whenever \( t_k \notin S_c/4 \), for a sufficiently large \( m^* \). For such a value of \( m^* \), select \( T = \frac{\Delta t}{m^*} \). Since by (28) the storage function is guaranteed to decrease between time instants \( t_k, k \in \mathbb{N} \), we claim that any trajectory starting from \( S_0 \) reaches \( S_{c/2} \) in finite time. For this were false, then we would have \( S(x(t_i), \dot{x}(t_i)) \leq S(x(t_0), \dot{x}(t_0)) \) and \( c\Delta t \rightarrow -\infty \) as \( i \rightarrow \infty \) which contradicts the positive definiteness of \( S \). Once \( [x(t^*), \dot{x}(t^*)] \in S_{c/2} \) then, by the choice of \( \Delta t \), \( [x(t^* + \Delta t), \dot{x}(t^* + \Delta t)] \in S_c \) the storage function is again decreasing and the trajectory of the system cannot leave \( S_c \).

Introducing sampling destroys asymptotic stability. With the help of the example of the double integrator, we can show that, indeed, there does not exist a constant sampling time, \( T > 0 \), under which the quantized and sampled system trajectories converge to the origin. Assume that the initial condition of the system is, again, at \([0, c]^T\). We can show that the largest delay that can be introduced in the switching while traversing the four quadrants of the phase plane must not exceed \( c/a \) if the system were to remain stable. Thus, a sampling period \( T \leq 2c/a \) is required for stability.

A delay of \( c/a \) is equivalent to a system emanating from the initial condition \([0, c]^T\) failing to switch when crossing the \( x_1 \)-axis of the phase plane, i.e., the change in \( [\dot{x}] \) is not detected at time \( t^* = c/a \). The trajectory evolves through the four quadrants of the phase plane while the control switches only twice, as tabulated below:

<table>
<thead>
<tr>
<th></th>
<th>( [x_1, x_2]^T )</th>
<th>( \dot{x}(\tau) )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>([0, c]^T)</td>
<td>(-a)</td>
<td>([0, c]^T)</td>
</tr>
<tr>
<td>( \pm \frac{\sqrt{c^2}}{a} )</td>
<td>(-a)</td>
<td>([0, c]^T)</td>
<td>(\pm \frac{\sqrt{c^2}}{a})</td>
</tr>
<tr>
<td>( \pm \frac{\sqrt{c^2}}{a} )</td>
<td>(a)</td>
<td>([0, c]^T)</td>
<td>(\pm \frac{\sqrt{c^2}}{a})</td>
</tr>
<tr>
<td>( \pm \frac{\sqrt{c^2}}{a} )</td>
<td>(-a)</td>
<td>([0, c]^T)</td>
<td>(\pm \frac{\sqrt{c^2}}{a})</td>
</tr>
</tbody>
</table>

As can be seen, the system trajectory is periodic, demonstrating that using \( T = \frac{\Delta t}{m^*} \) produces trajectories that stay in the complement of the level set \( S_c \). Asymptotic stabilization hence requires that \( T \to 0 \).

VI. CONCLUSION

Under the condition stipulated by Proposition 1, a continuous 2\textsuperscript{nd} order system is asymptotically stabilized by quantized feedback for any quantum size, provided that the gains obey the passivity conditions of Proposition 2. This result is surprising since it is usually thought that stabilization is only possible within the resolution of the sensors. For set-point control, the system can be controlled to converge exactly to any quantization boundary. Considering quantization alone is akin to assuming that the sampling rate is sufficiently fast to ignore its effects, otherwise, only practical stabilization can be achieved and most systems will enter a limit cycle in the neighborhood of the origin.

This theory need to be extended to the quantized and sampled control of any \( \mathcal{L}_1 \) system. We also plan to relate it to practical problems such as achieving accurate control in virtual reality haptic systems [11], as well as to lift the assumption that velocity sensors are available.

REFERENCES