A LINEAR APPROACH FOR MULTIPLE-POINT IMPACT IN MULTIBODY SYSTEMS

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Keywords: multiple-point impact, contact dynamics

Abstract. *The dynamical analysis of multibody systems undergoing simultaneous multiple-point collisions is a relevant problem in various fields, such as robotics and biomechanics. Different approaches to study collisions can be found in literature, going from totally impulsive ones (dealing only with the pre- and post- impact states and assuming that the system configuration is unchanged) to totally continuous ones (where the time integration of the equations of motion is done and the system configuration changes throughout the collision interval). Simultaneity cannot be treated properly in impulsive approaches because the mathematical formulation shows indetermination. These approaches try to overcome this drawback by defining single-point collision sequences (not necessarily realistic) and the final results are often sequence-dependent. Continuous methods are better suited to deal with simultaneity, but they often result into complicated models. This article proposes a simple hybrid linear approach based on a vibrational dynamical model. The dynamics at the colliding points is simulated through linear stiff springs undergoing very small deformations and thus generating a vibratory behavior of the system. The overall system configuration is assumed to be constant as far as the inertia matrix is concerned. The collision end corresponds to a zero spring force. Only planar application cases will be presented, but the approach is suitable for 3D multiple-point smooth collisions in multibody systems with perfect constraints. Though only the case of perfectly elastic collisions will be shown, the methodology can be extended to collisions showing any degree of inelasticity.*

1   INTRODUCTION

The dynamics of multiple-point collisions in multibody systems is a topic still under discussion. The usual approaches found in literature to deal with it can be roughly classified into impulsive approaches and continuous ones [1,2]. Impulsive methods neglect the collision duration; the post-impact mechanical state is obtained from the pre-impact one assuming that the system configuration is constant throughout the collision interval. These methods cannot deal in general with simultaneous collisions because the set of equations governing the process exhibits indetermination. Under the hypothesis of perfect rigid bodies, which is the simplest model, small perturbations on the impact configuration may result in different sequences of single point collisions yielding different end conditions. If finite normal stiffness is assumed at the impact points, such sensitivity to initial conditions is associated to that of the evolution of the normal forces. Most works assume either a collision sequence with no overlapping (sequence of single point collisions) [3], or a total simultaneity of uncoupled independent impacts [4], and use Newton's or Poisson's restitution coefficients modified according to energy criteria. Other authors choose a stochastic description of the problem and represent the solution as a random vector [5]. However, the final results are often sequence-dependent (if assuming a collision chronology), and not always realistic.

In pure continuous methods, the contact surfaces are modeled through nonlinear springs and dampers, and the equations of motion are integrated during the impact time interval to obtain the post-impact mechanical state. They are far more demanding than the impulsive ones from the computational point of view, and the final results may be more realistic. The advantage of continuous approaches as compared to impulsive ones is that indetermination is avoided.

This article proposes a simple linear approach retaining the high sensitivity to initial conditions without assuming any particular collision sequence and allowing any degree of overlapping. This approach is neither impulsive nor perfectly continuous but somehow hybrid, as it assumes constant configuration but vibrational continuous dynamics with a convenient time scale. At each impact point, the contact between solids is modeled through a finite linear normal stiffness (high enough to assume constant configuration throughout the process). The set of actually colliding points may change along the process, and consequently the number of stiff springs. The linear springs are only compatible with perfectly elastic collisions. In order to simulate inelastic ones, a suitable friction element should be coupled to the springs.

At every time instant, that set defines a collection of vibration modes which allow to keep track of the normal velocities and displacements at the impact points in a simple analytical way. The inertia matrix in this vibrational formulation, which is constant, is obtained through a physically meaningful decomposition of the kinetic energy of the system into two components: that associated with the motion in the normal direction of all the impact points ("constrained motion"), and that associated with the motion with zero normal velocities ("admissible motion").

The approach is presented for systems in which the colliding points do no present redundancy. This leads to a number of vibrational modes (associated with the number of colliding points) lower or equal to the system's number of Degrees Of Freedom (DOF). A more general procedure including redundancy has also been worked out, and will be addressed in upcoming publications.

Two application examples will be presented, both corresponding to planar motions. The first one consists on a rigid rod colliding with a smooth surface. Special attention will be paid to the sensitivity of the system to the collision sequence. The second example is a simple multibody model of an individual walking with crutches. The contact of the tip crutch with the ground starts with a collision which can be studied successfully by means of this vibratory approach.

2 NORMAL IMPULSES AND INCREMENTAL CHANGES OF THE NORMAL VELOCITIES

Let's consider a n DOF multibody system, described by the n vector of generalized velocities $\dot{q}$, with m possible and simultaneously colliding points whose normal velocities are related to the generalized velocities through a $m \times n$ matrix of kinematic coefficients, $v_n = A\dot{q}$. We will restrict our study to the case where the m normal velocities v_n are independent and, consequently, $\operatorname{rank}(A) = m$. The system percussive dynamics Lagrangian equations are

$$\mathbf{M}\Delta\dot{q} = \boldsymbol{\Pi}_n, \tag{1}$$

where $\mathbf{M}$ is the $n \times n$ inertia matrix for the impact configuration, and $\boldsymbol{\Pi}_n$ is the vector of generalized normal impulses. The n vector $\boldsymbol{\Pi}_n$ is related to the m vector $\boldsymbol{P}_n$ of normal impulses at the impact points through $\boldsymbol{\Pi}_n = \mathbf{A}^T \boldsymbol{P}_n$. The incremental changes $\Delta\dot{q}_{imp}$ and Δv_n are related to the normal impulses $\boldsymbol{P}_n$ through

$$\Delta\dot{q}_{imp} = \mathbf{M}^{-1}\mathbf{A}^T \boldsymbol{P}_n \quad \text{and} \quad \Delta v_n = \mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T \boldsymbol{P}_n. \tag{2}$$

As $\operatorname{rank}(A) = m$, $\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T$ is a symmetrical and positive definite $m \times m$ matrix. Accordingly, there exists a biunique relationship between Δv_n and $\boldsymbol{P}_n$. Thus, if Δv_n are known, $\boldsymbol{P}_n$ and the associated $\Delta\dot{q}_{imp}$ can be calculated as

$$\boldsymbol{P}_n = \left(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T\right)^{-1}\Delta v_n \quad \text{and} \quad \Delta\dot{q}_{imp} = \mathbf{M}^{-1}\mathbf{A}^T\left(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T\right)^{-1}\Delta v_n. \tag{3}$$

The time evolution of vector Δv_n associated with the multiple-point impact can be conveniently estimated by means of a linear model for the stiffness at the collision points. The nonlinear Hertz model would lead to a time consuming integration process, while the linear approach leads to a problem of modal superposition. The actual contact points may change along the impact process, therefore the stiffness matrix and, consequently, the eigenfrequencies and eigenmodes, will also change. However, for each time interval where the set of contact points is invariant, the evolution of the normal velocities v_n and displacements $\boldsymbol{\delta}_n$ can be analytically obtained by means of a modal superposition. If $\boldsymbol{\delta}_n = 0$ is associated with contact without compression at the impact point, the set of actual colliding points changes whenever:
- $\delta_{nj} = 0$ and $v_{nj} > 0$ (end of collision at point j)
- $\delta_{ni} = 0$ and $v_{ni} < 0$ (starting collision at point i)

3 FORMULATION OF THE LINEAR MODEL

Substitution of $\Delta v_n = \mathbf{A}\,\Delta\dot{q}$ in Eq. (3) leads to

$$\Delta\dot{q}_{imp} = \mathbf{M}^{-1}\mathbf{A}^T\left(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T\right)^{-1}\mathbf{A}\,\Delta\dot{q} \equiv \mathbf{H}_c\Delta\dot{q}. \tag{4}$$

3

Matrix $\mathbf{H}_c$ is a well-known matrix associated with the decomposition of the kinetic energy for unilaterally-constrained multibody systems [6]. Matrices $\mathbf{H}_c$ and $(\mathbf{I} - \mathbf{H}_c)$ (with $\mathbf{I}$ = identity matrix) decompose the generalized velocity vector $\dot{\boldsymbol{q}}$ into two components, $\dot{\boldsymbol{q}} = \boldsymbol{u}_c + \boldsymbol{u}_a$, with $\boldsymbol{u}_c = \mathbf{H}_c \dot{\boldsymbol{q}}$ and $\boldsymbol{u}_a = (\mathbf{I} - \mathbf{H}_c)\dot{\boldsymbol{q}}$, associated with the spaces of constrained and admissible motions respectively. The kinetic energy then separates into kinetic energy associated with the constrained motion,

$$T_c = (1/2)\,\boldsymbol{u}_c^T \mathbf{M} \boldsymbol{u}_c = (1/2)\,\boldsymbol{v}_n^T \left(\mathbf{A} \mathbf{M}^{-1} \mathbf{A}^T \right)^{-1} \boldsymbol{v}_n, \tag{5}$$

and that associated with the admissible motion, $T_a = (1/2)\,\boldsymbol{u}_a^T \mathbf{M} \boldsymbol{u}_a$. According to Eq. (5), the reduced matrix $\mathbf{M}_n \equiv \left(\mathbf{A} \mathbf{M}^{-1} \mathbf{A}^T \right)^{-1}$ is the $m \times m$ inertia matrix associated with the normal velocities $\boldsymbol{v}_n$. If elastic constraints with stiffness constant k_j are assumed at the collision points, a m DOF linear vibration problem may be formulated by means of the inertia matrix $\mathbf{M}_n$ and the stiffness matrix $\mathbf{K}_n^*$ with elements $k_j^* = k_j$ if there is contact at the collision point j, and $k_j^* = 0$ otherwise. Eigenvalues and eigenvectors of the dynamical matrix $\mathbf{D} \equiv \mathbf{M}_n^{-1} \mathbf{K}_n^*$ define the eigenfrequencies and eigenmodes of the vibration problem. The evolution of $\boldsymbol{v}_n(t)$ and $\boldsymbol{\delta}_n(t)$ can be obtained from the initial conditions through modal superposition. Whenever there is a change in the set of contact points, $\mathbf{K}_n^* \, \mathbf{K}_n^*$ is changed accordingly, and the eigenmodes and eigenvalues are recalculated. The initial conditions for the new phase are the final ones of the preceding phase. The collision problem is over when $v_{nj} > 0$ and $\delta_{nj} \geq 0$ everywhere. The total change of the system generalized velocities $\Delta \dot{\boldsymbol{q}}_{imp}$, can be obtained from that of the normal velocities $\Delta \boldsymbol{v}_n = \boldsymbol{v}_n(t_{end}) - \boldsymbol{v}_n(0)$ through Eq. (3).

4 APPLICATION EXAMPLES

Two application examples with planar motion have been considered: a single rigid body system and a multibody one. The ground stiffness has been taken $10^8\ N/m$ for the first one, and $10^9\ N/m$ for the second one. Different initial states have been explored in order to illustrate the sensitivity to the initial conditions, characteristic of collision problems.

4.1 Example I: Single rigid body

The first example consists on a two-point impact of a rod on a fixed ground, Fig. (1). The mass is concentrated on the first half of its length, and the colliding points P and Q will be located at the rod ends. As the two normal velocities are independent, there is no redundancy.
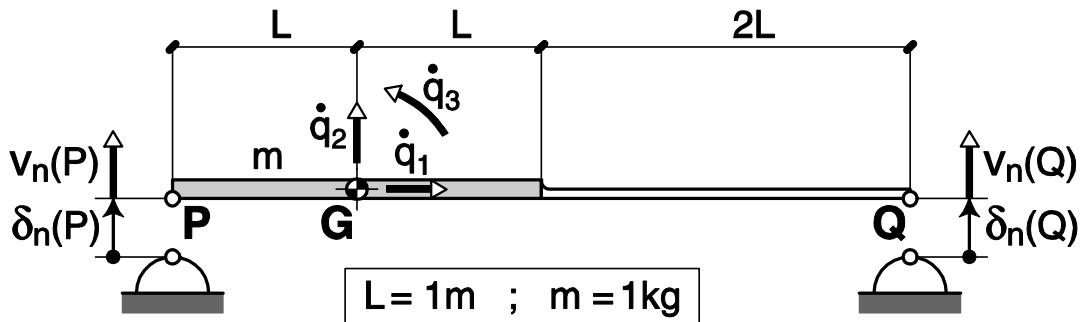

Figure 1: Rod colliding with a fixed ground.

We have explored two different sets of initial velocities (pure vertical downwards translation, and downwards translation plus clockwise rotation) while keeping a same initial configuration (horizontal). In order to explore the system sensitivity to different collision sequences, different initial normal displacements (a few micrometers) of the colliding points, $\delta_n^-(P)$ and $\delta_n^-(Q)$, have been considered in order to force different contact forces evolution.

For each set of initial velocities, two different cases have been simulated: simultaneous collision $\left(\delta_n^-(P) = \delta_n^-(Q) = 0\right)$ and first collision at Q $\left(\delta_n^-(P) > 0, \delta_n^-(Q) = 0\right)$. The plots show the evolution of the P and Q normal displacements (upper left), kinetic energy (lower left), translation DOF (upper right) and rotation DOF (lower right). As no dissipation has been considered, all cases show a final kinetic energy equal to the initial one. However, its evolution is different according to the collision phases. The P and Q final velocities and the rod DOF show not only different evolutions but also different final values depending on the sequence. Table 1 contains all this information for the different cases plotted in Figs. (2) to (5).

Fig.	Initial conditions		Final state
	Normal displacements $\left\{\delta_n^-(P), \delta_n^-(Q)\right\}$	Generalized velocities $\left\{\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3\right\}$	P, Q normal velocities $\left\{v_n^+(P), v_n^+(Q)\right\}$
2	$\{0 \quad 0\}$	$\{0 \quad -.1(m/s) \quad 0\}$	$\{113.10 \quad 59.49\}(mm/s)$
3	$\{35\,\mu m \quad 0\}$	$\{0 \quad -.1(m/s) \quad 0\}$	$\{121.54 \quad 32.04\}(mm/s)$
4	$\{0 \quad 0\}$	$\{0 \quad -.1(m/s) \quad -.1(rad/s)\}$	$\{62.99 \quad 258.86\}(mm/s)$
5	$\{25\,\mu m \quad 0\}$	$\{0 \quad -.1(m/s) \quad -.1(rad/s)\}$	$\{65.31 \quad 253.06\}(mm/s)$

Table 1: Initial conditions and final velocities of the rod colliding points.
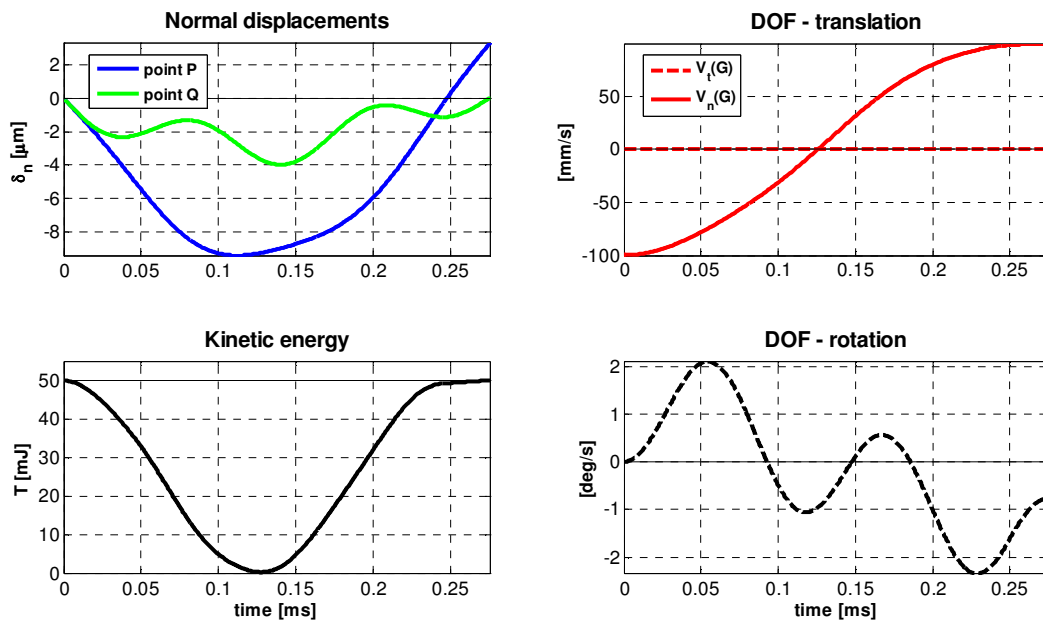

Figure 2: Evolution of $\delta_n(t)$ and $T(t)$ (left) and of the three DOF (right) in a two-point impact of a rod for the case of pure downwards translation. The collision starts simultaneously at both points.
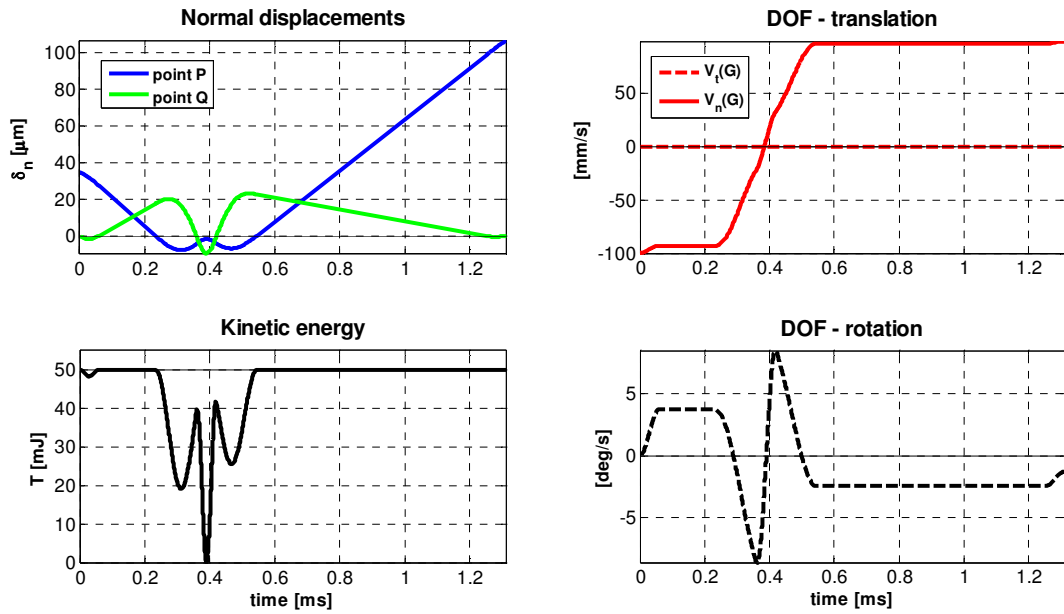
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Figure 3: Evolution of $\delta_n(t)$ and $T(t)$ (left) and of the three DOF (right) in a two-point impact of a rod for the case of pure downwards translation. The collision at point Q starts before that at point P.
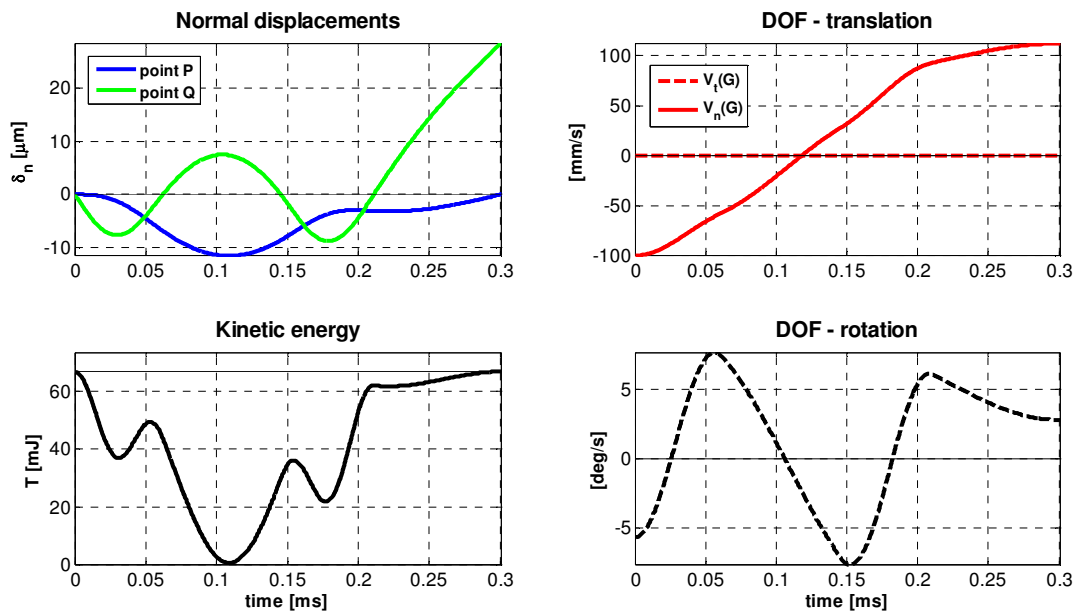

Figure 4: Evolution of $\delta_n(t)$ and $T(t)$ (left) and of the three DOF (right) in a two-point impact of a rod for the case of downwards translation plus clockwise rotation. The collision starts simultaneously at both points.
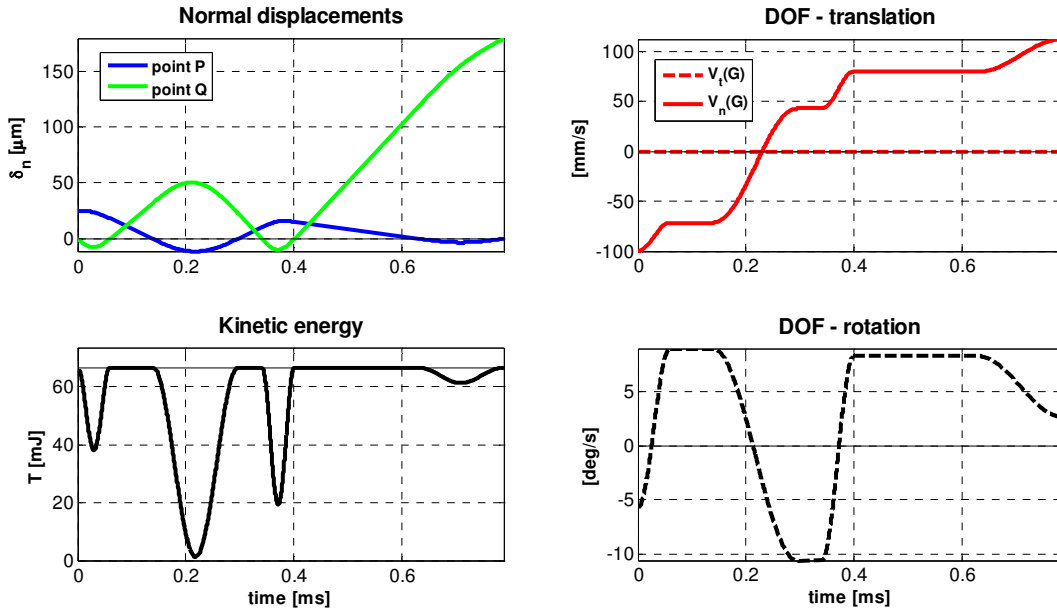
Figure 5: Evolution of $\delta_n(t)$ and $T(t)$ (left) and of the three DOF (right) in a two-point impact of a rod for the case of downwards translation plus clockwise rotation. Collisions show no overlapping.

All results show the dynamic coupling between the two contact points in all cases. Fig. (2) corresponds to simultaneous collisions showing impact overlapping in approximately 90% of the impact interval, thus leading to a final state that would never be obtained if assuming sequential single point impacts. Fig. (3) shows a first short single point collision phase, where Q is the colliding point, followed by a free motion (that is, without ground contact). The coupling between points P and Q is responsible for a second single point collision starting around $t = 0.22\ ms$. However, point Q collides again before P finishes the decompression phase, and so overlapping appears. Though the final phase is again a single point collision, the overlapping appearing around $t = 0.35\ ms$ is responsible for a final state different from the one that would be obtained without any overlapping.

Fig. (4) shows again a simultaneous two-point collision, but unlike the case in Fig. (2) there is a middle phase without overlapping (roughly between $t = 0.06\ ms$ and $t = 0.14\ ms$). Keeping the same initial velocities but perturbing the initial normal displacements, it is possible to obtain a clearly single-point collision sequence as that shown in Fig. (5).

As in that case we are dealing with single-point collisions on a smooth surface, the results could be obtained analytically by means of Newton's rule with a restitution coefficient equal to one. Thus, for any collision in the sequence, $v_n^+ = v_n^-$ for the colliding point, though in general $v_n^+ \neq v_n^-$ for the non colliding point.

Fig. (6) shows the normal displacements and normal separation velocities of points P and Q for the same initial velocities as those of case in Fig. (5). The analytical calculation through the corresponding algebraic equations yields the same set of mechanical states for that collision sequence.

7
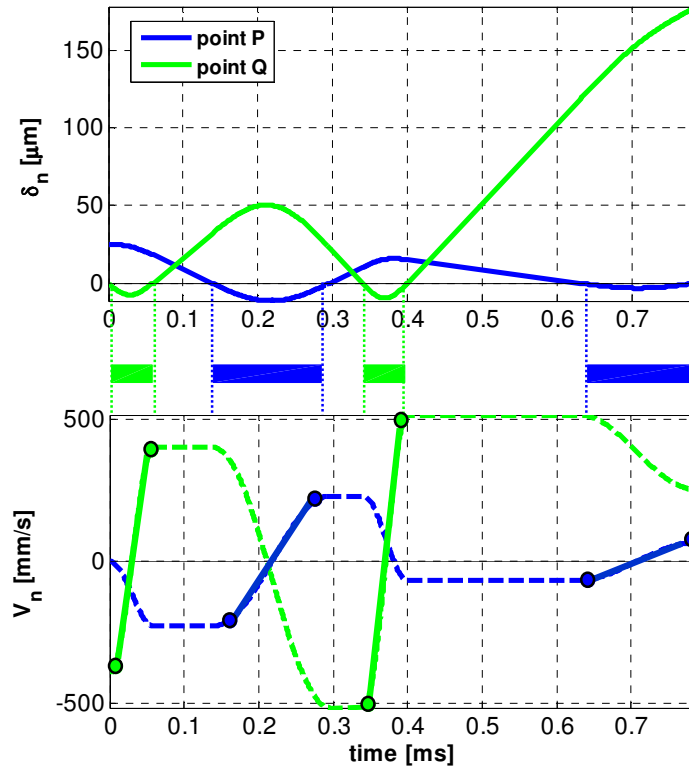
Figure 6: Evolution of $\delta_n(t)$ and $v_n(t)$ in the two-point impact of the rod in Fig. (5).

4.2 Example II: Multibody system

A simple planar model of a subject with crutches is shown in Fig. (7). It is composed of four segments (legs, torso, upper arms, and arms plus crutches) linked by revolute joints, modeling the hip, shoulder and elbow joints. Coordinates q_1 and q_2 indicate the position of the feet, coordinate q_3 denotes the absolute orientation of the legs, q_4 is the relative angle between torso and legs, q_5 is the relative angle between upper arms and torso, and q_6 is the relative angle between crutches and upper arms. All the angular coordinates are clockwise orientated. The anthropometric parameters are the ones for a subject whose total mass is 70 kg and the height is 1.75 m according to [7]. The crutches have a mass of 1.2 kg and are 1 m long These parameters are summarized in Table 2.

	Legs	Torso	Upper Arms	Arms+crutches
m (kg)	22.54	40.46	3.92	4.28
I_G (kg·m^2)	2.07	2.66	0.044	0.433
l (m)	0.93	0.51	0.33	1.25
a (m)	0.51	0.34	0.14	0.33

Table 2: Anthropometric parameters of the model of the individual with crutches.

8

In real crutch walking, the post-impact normal velocity of the crutch tip has to be zero. However, this condition will never be attained with the present approach as neither dissipation nor friction have been included. The purpose of the example is to explore the validity of our approach for a multibody system with planar motion rather than study realistic crutch walking.

Two different sets of initial velocities have been explored but only one initial configuration (leading to initial ground contact at feet and crutch tip) has been considered. This information is summarized in Table 3. As in the previous application example, the plotted variables in Figs. (8) and (9) are the evolution of the feet and crutch tip normal displacements (upper left), the kinetic energy (lower left), the translation DOF (upper right) and the rotation DOF (lower right). The absence of dissipation leads again to a conservation of the kinetic energy.
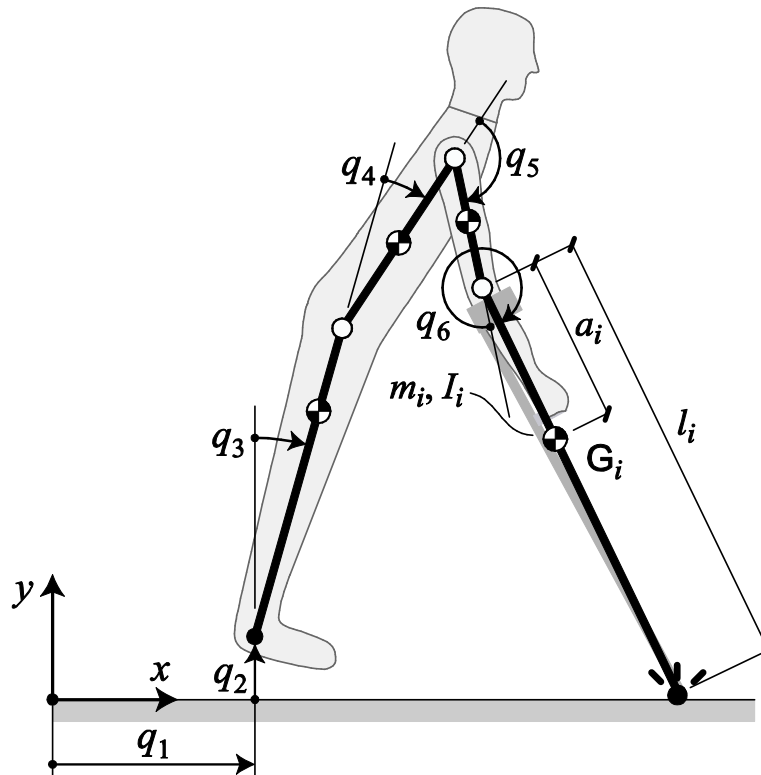

Figure 7: Planar model of an individual walking with crutches.

Fig.	Initial configuration	Initial velocities
8	$q^- = \{0 \quad 0 \quad 15^\circ \quad 10^\circ \quad 150^\circ \quad -20^\circ\}$	$\dot{q}^- = \{0 \quad 0 \quad 2(rad/s) \quad 0 \quad 0 \quad 0\}$
9		$\dot{q}^- = \{0 \quad -0.5(m/s) \quad 1(rad/s) \quad 0 \quad 0 \quad 0\}$

Table 3: Initial states explored for the individual with crutches.

Fig. (8) shows a single-point collision (crutch tip) case. The initial state is responsible for a low positive value for the feet normal separation velocity derivative, so a feet upward motion $\left(v_n^-(\text{feet}) > 0\right)$ is immediately started. This upwards motion is monotonous throughout the impact interval. As there is only one spring undergoing compression, the evolution of the kinetic energy follows a sinusoidal evolution.

9

Normal displacements

DOF - translation

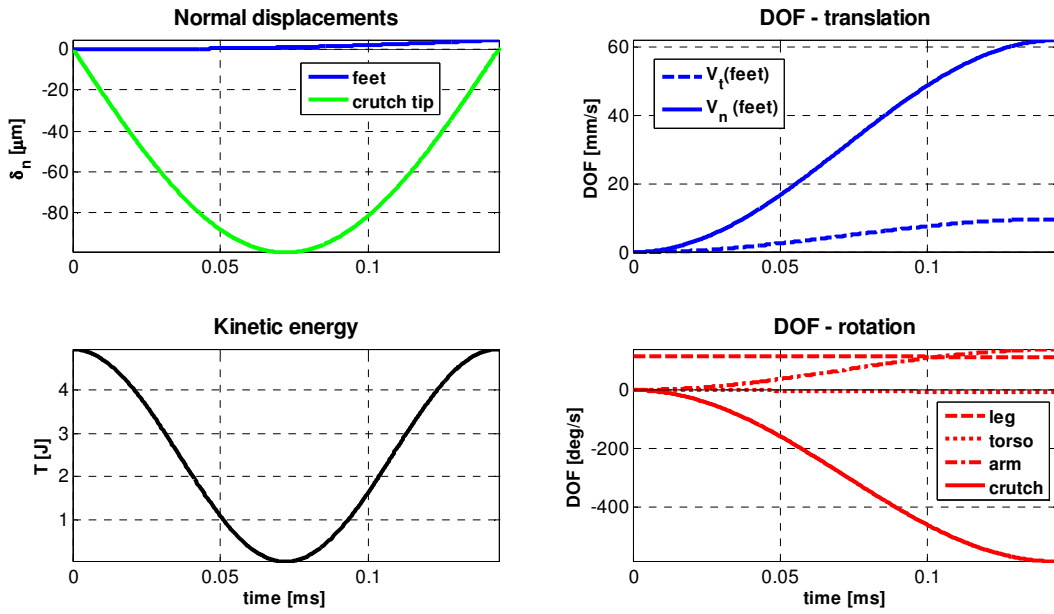**Kinetic energy**

DOF - rotation

Figure 8: Evolution of $\delta_n(t)$ and $T(t)$ and of the system's DOF in a simple crutch walking model. The initial state leads to a single-point collision.

Fig. (8) shows a single-point collision (crutch tip) case. The initial state is responsible for a low positive value for the feet normal separation velocity derivative, so a feet upward motion $\left(v_n^-(\text{feet}) > 0\right)$ is immediately started. This upwards motion is monotonous throughout the impact interval. As there is only one spring undergoing compression, the evolution of the kinetic energy follows a sinusoidal evolution.

Normal displacements

DOF - translation

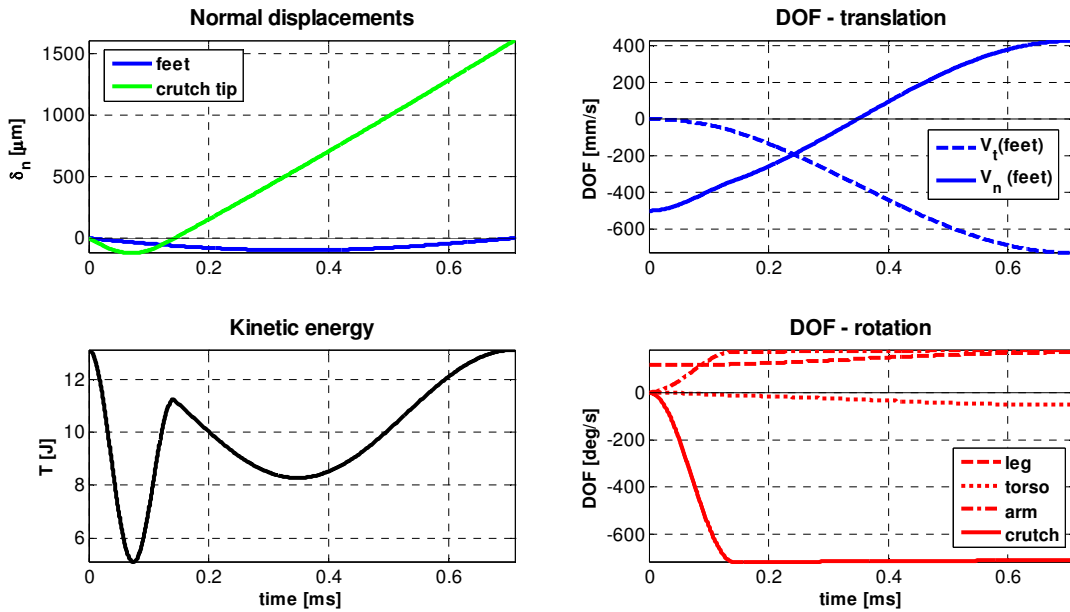**Kinetic energy**

DOF - rotation

Figure 9: Evolution of $\delta_n(t)$ and $T(t)$ and of the system's DOF in a simple crutch walking model. The initial state leads to a first two-point simultaneous collision, and a second phase consisting on a single-point collision.

10

In Fig. (9) the initial state results in $v_n^-\left(\text{crutch}\right)<0$ and $v_n^-\left(\text{feet}\right)<0$, so a two-point collision takes place. The first minimum in the kinetic energy plot is mainly associated with both feet and crutch compression. The second wide minimum is associated exclusively to the feet compression.

Fig.	Initial configuration	Initial generalized velocities
8	$q^-=\left\{0 \quad 0 \quad 15^o \quad 10^o \quad 150^o \quad -20^o\right\}$	$\dot{q}^-=\left\{0 \quad 0 \quad 2\left(rad/s\right) \quad 0 \quad 0 \quad 0\right\}$
9		$\dot{q}^-=\left\{0 \quad -0.5\left(m/s\right) \quad 1\left(rad/s\right) \quad 0 \quad 0 \quad 0\right\}$

Table 3: Initial states explored for the individual with crutches.

5 CONCLUSIONS

A simple linear vibrational approach has been presented to study smooth multiple-point impacts in multibody systems with perfect constraints and 3D motion. It has been presented for perfectly elastic collisions for the sake of simplicity, but it can be easily extended to collisions showing any degree of inelasticity. For the same reason, only the case of colliding points without redundancy has been considered, but the method can be extended to redundant colliding points.

The approach assumes a constant system configuration as far as the system inertia matrix is concerned, and so in this respect it is close to impulsive approaches. Nevertheless, it assumes continuous vibrational linear dynamics at the colliding points, with a convenient time scale in order to be consistent with the system constant configuration assumption. A characteristic feature of the approach is the use of a reduced inertia matrix associated with the possible simultaneous colliding points. Its dimension is equal to the number of these points, regardless the number of DOF exhibited by the system. That matrix is obtained by means of a physically meaningful decomposition of the system kinetic energy into that associated with the motion in the normal directions at the colliding points, and that associated with motion compatible with zero normal velocities.

At each actually colliding point, the contact between bodies is modeled through a linear normal stiffness, high enough to guarantee the constant inertia matrix assumption. The set of actual colliding points may change along the process, and consequently the set of stiff springs to be considered and the resulting stiffness matrix used in the vibrational formulation. At every time, that set defines a collection of vibration modes which allow to keep track, in a simple analytical way, of the normal velocities and displacements of the full set of possible colliding points.

As the normal forces at the colliding points are formulated from the normal displacements, the approach retains the high sensitivity of multiple-point collisions to initial conditions. Small perturbations of the initial normal displacements may lead to a quite different evolution of the normal forces and, consequently, of the system final velocities.

The two application examples show the approach capacity to retain the high sensitivity to initial conditions without assuming particular sequences of single point collisions, as is usually done in impulsive approaches, and without the time consuming integration process that

11

would require the use of the nonlinear contact stiffness Hertz model. For systems with a high number of DOF, the presented approach is also less time consuming than that of a vibratory linear model directly associated with the system original (without reduction) inertia matrix.

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