Whitening of Filtering Error Using Nonuniform Sampling

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Abstract

When sampling audio signals for storage or coding prior to transmission, one passes the signal through an anti-aliasing filter. This filter prevents aliasing error when the signal is reconstructed, but has the side-effect of chopping off an important part of the spectrum of the signal. In speech applications this results in a muffling of the sound, while in musical applications it removes the "sparkle" or brilliance of the sound. This removal of a portion of the spectrum is referred to as "filtering error". In this paper we show that, by using a nonuniform sample sequence, with an appropriate anti-aliasing filter, one can whiten the spectrum of the filtering error as well as allow more high frequency information to be captured. The idea is that, for some audio applications, the added random noise will be less objectionable than the muffling of the sound produced by standard sampling.

1 Filtering Error in Sampled Data Systems

It is well known that in order to uniformly sample and reconstruct a signal without error, the signal cannot have any frequency components greater than one half the sampling rate (referred to as the Nyquist frequency). If this constraint is not satisfied, the sampling process will produce an overlapping of repetitions of the signal spectrum, resulting in the so-called *aliasing error*.

In practice one does not have control over the signals that are to be sampled and, therefore, signals to be sampled often do contain frequencies greater than the Nyquist frequency. In order to handle these sorts of signals, and avoid aliasing error, one must remove the frequency components above the Nyquist frequency with a lowpass filter. This filter is called an *anti-aliasing filter*. The signal that is reconstructed from the samples is identical to the antialiased version of input signal. For most purposes this is sufficient. There are many applications, however, for which the effect of the anti-aliasing filter is deleterious. For example, the excessive lowpass filtering of speech encountered in low bandwidth communication channels such as telephone lines, causes a muffling of the sound, which reduces intelligibility. Another example is the electronic music "sampler". These devices store digitized samples of the sounds made by musical (or sometimes non-musical) instruments which are then played back under performer control. In order to economize on the amount of memory used in the systems the product of the sample length and the sampling rate is kept as low as possible. For sounds with a long decay tail and significant energy in high frequency components, such as those of cymbals, this constraint results in either truncation of the sound or a muffling, or loss of brilliance, of the sound.

The error in the reconstructed signal resulting from the use of anti-aliasing filters is known as filtering error. Anti-aliasing has the drawback (for applications whereing humans will be listening to the result) that the filtering error is correlated with the input signal. This plays a large part in the objectionableness of the resulting sound to human ears. The objectionableness of aliasing error is likewise due to the correlation of the error with the signal. In contrast, error mechanisms which result in random, wide band, errors which are uncorrelated with the input signal are perceived as being less objectionable by humans. The question which arises, then, is: Is there a sampling and reconstruction scheme whereby any errors that arise are uncorrelated with the signal. That is, we want to come up with a

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way to *whiten* the filtering error (or perhaps the aliasing error if there is any) induced by the reconstruction process.

In the remainder of the paper we propose a method which does whiten the spectrum of the filtering error of a sampling and reconstruction process. The filtering error using this method, although randomized, may also be greater in magnitude than for the standard sampling and reconstruction process. Whether the noise level is increased unacceptably depends on the closeness of the matching of the signals to the sampling scheme.

2 Nonuniform Sampling and Reconstruction of Signals

The sample sequence used in a signal sampling and reconstruction system need not be uniform. Clark et al [1] have shown that one can reconstruct a suitable class of signals $f(t)$ *exactly* from their nonuniformly spaced samples $\{f(t_n)\}$ with the following reconstruction formula:

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)\mathrm{sinc}(\gamma(t) - n) \qquad (1)$$

where $\gamma(nT) = t_n$. The warping or distortion function $\gamma(t)$ is not uniquely determined by the constraint imposed on it by the sample sequence. The space of signals that is comprised of those signals for which the above equation holds is defined by the particular choice of $\gamma(t)$ that is made.

If a signal is not in the space of signals for which the above equation holds there will neccessarily be an error in the reconstruction of the signal. This error is analogous to the aliasing error of the uniform sampling and reconstruction approach. In order to avoid this "aliasing" error it is expected that we need to apply some sort of "anti-aliasing" filter to the signal. It is not immediately obvious what form this anti-aliasing operation is to take.

3 Anti-aliasing as Projection

Let us examine more closely the activity of the anti-aliasing filter. To allow a clear generalization of anti-aliasing to include non-uniform sampling and reconstruction techniques let us look at filters as operators (in the mathematical sense).

Let us first define the following signal spaces. Let $\mathcal{B}_{\omega_0}$ be the subspace of $\mathbf{L}^2(\mathbf{R})$ consisting of signals that are bandlimited to a frequency of ω_0. Let $\mathcal{B}_\Gamma$ be the space of signals that satisfy equation (1) above, for a given γ.

Let $\mathbf{G}$ be the bandlimit operator, i.e. the operator that takes functions in $\mathbf{L}^2(\mathbf{R})$ into the subspace $\mathcal{B}_{\omega_0}$. That is,

$$\mathbf{G}f(t) = f(t) * \mathrm{sinc}(\omega_0 t) \qquad (2)$$

where the $*$ indicates the convolution operation. $\mathbf{G}$ is easily seen to be a *projection* operator, since $\mathbf{GG} = \mathbf{G}$. Thus the standard anti-aliasing filter for uniform sampling can be thought of as performing a projection into the space of bandlimited signals (with bandlimit ω_0 of the filter). If $f(t)$ is bandlimited (to ω_0) then $\mathbf{G}f(t) = f(t)$, and $f(t)$ is an eigenfunction (with unit eigenvalue) of the bandlimit operator. In general we can say that bandlimited signals are those signals that are eigenfunctions (with unit eigenvalue) of bandlimit operators.

Now let us consider the design of an anti-aliasing filter for the case of non-uniform sampling. From the projection operator point of view presented above we see that the anti-aliasing filter for nonuniform sampling is merely the projection operator related to the space $\mathcal{B}_\Gamma$. Let us call this operator $\hat{\mathbf{G}}$. The signals which satisfy equation (1) will be eigenfunctions (with unit eigenvalue) of this projection operator. The form of this projection operator has been derived by Shlomot and Zeevi [4] and is given by

$$\hat{\mathbf{G}}f(t) = \int_{-\infty}^{\infty} f(\gamma^{-1}(\tau))\mathrm{sinc}(\gamma(t) - \tau)d\tau \qquad (3)$$

Let Γ be the time warping operator defined by $\Gamma f(t) = f(\gamma(t))$, with inverse Γ^{-1} defined by $\Gamma^{-1}f(t) = f(\gamma^{-1}(t))$. Thus, using the above equation we can write

$$\hat{\mathbf{G}}\Gamma f(t) = \int_{-\infty}^{\infty} f(\tau)\mathrm{sinc}(\gamma(t) - \tau)d\tau \qquad (4)$$

then we can write

$$\Gamma^{-1}\hat{\mathbf{G}}\Gamma f(t) = \int_{-\infty}^{\infty} f(\tau)\mathrm{sinc}(t - \tau)d\tau = \mathbf{G}f(t) \qquad (5)$$

Hence we can see that

$$\hat{\mathbf{G}}f(t) = \Gamma\mathbf{G}\Gamma^{-1}f(t) \qquad (6)$$

Note that $\hat{\mathbf{G}}\hat{\mathbf{G}} = \hat{\mathbf{G}}$, and hence $\hat{\mathbf{G}}$ is a projection operator. The projection operation for the nonuniform sampling/reconstruction scheme can be seen to consist of a time warping operation, followed by a bandlimiting operation followed by an unwarping operation. From equation (3) one can interpret the nonuniform antialiasing operation as filtering with a *time varying* lowpass filter.

4 Filtering Error Spectrum

The filtering error will be zero for signals which are eigenfunctions, corresponding to the unit eigenvalue, of the projection operator used in the anti-aliasing operation. In the case of uniform sampling these eigenfunctions are signals which are bandlimited to the anti-aliasing filter cutoff frequency, ω_0. Signals which are bandlimited, but to a frequency higher than the Nyquist frequency, will incur a nonzero filtering error, as the frequency components above the Nyquist frequency will be removed by the anti-aliasing filter. Signals that are non-bandlimited, such as FM (frequency modulated) signals, will always incur some level of filtering error, as they will always have some frequency components above the Nyquist frequency.

The case of nonuniform sampling is somewhat more complicated. Here, the eigenfunctions of the projection operator are generically (and perhaps always) non-bandlimited [3]. This can be seen as follows. Let $f(t)$ be an eigenfunction of the anti-aliasing projection operator defined by a time-warping Γ. Then we have that

$$\hat{\mathbf{G}} f(t) = (\Gamma G \Gamma^{-1}) f = f$$

or

$$G(\Gamma^{-1} f) = \Gamma^{-1} f$$

This means that, for $f(t)$ to be bandlimited, there must exist a bandlimited signal $h(t) = \mathbf{G}(\Gamma^{-1} f(t))$ for which $f(t) = \Gamma h(t)$ is bandlimited. The action of the Γ operator is equivalent [1] to a phase modulation. It can be shown [3] that phase modulated signals are generically non-bandlimited.

One of the implications of the above is that, in the case of nonuniform sampling, there is a class of non-bandlimited signals (the eigenfunctions of the anti-aliasing operator $\Gamma G \Gamma^{-1}$ for which there is no filtering error. Unlike the uniform sampling case, the nonuniform anti-aliasing filter passes through arbitrarily high frequencies. The phase structure of these signals must be precisely matched to the time warping function $\gamma(t)$ for there to be no filtering error. In general, functions that incur no filtering error are what Clark [2] refers to as Generalized Phase Modulated Signals. These are signals which are created by phase modulating bandlimited signals with monotonic signals. For example, if $g(t)$ is bandlimited, and $\gamma(t)$ is monotonic, then $f(t) = g(\gamma(t))$ is a Generalized Phase Modulated Signal, and, in addition, will incur no filtering error when sampled with a sequence that satisfies $\gamma(n) = t_n$, and reconstructed with a time varying filter $\hat{\mathbf{G}}$.

Even for signals that are not eigenvalues of the anti-aliasing operator, the nonuniform approach may be useful. This is due to the fact that some high frequency components of the signal are passed through. The antialiased signal can be thought of as a superposition of eigenfunctions of the anti-aliasing operator. Since these are all generically bandlimited, there will be components of the reconstructed signal at all frequencies. Clearly some of these will be noise, or artifacts. For example, if our input signal is bandlimited, then all of the components of the reconstructed signal above the Nyquist frequency will be artifacts of the reconstruction process. In general it expected that some high frequency (above the nominal Nyquist rate, as defined by the average sampling rate) information will be passed through. There will be, however, phase and magnitude distortion of these high frequency components. If the sampling sequence is somewhat random (e.g. a random perturbation or deviation from a uniform sequence) then these distortions will be random as well. Thus we will have acheived our goal or randomizing the filtering error in a sampling and reconstruction scheme.

As an illustration of the ability of the anti-aliasing filters for non-uniform sample sequence to pass through some of the frequency components above the Nyquist frequency we will look at the effect of the anti-aliasing filters on a square wave signal. In figures 1 through 5 we show the effect of anti-aliasing on a square wave signal for the case of uniform sampling and for the case of random perturbations (uniformly distributed over a range of $\pm 0.3T$ where T is the nominal sampling period) of a uniform sampling sequence. The square wave has a fundamental period of 100 time units, while the effective sampling period (for the case of uniform sampling is one time unit (i.e. the frequency of the square wave is 1/100 cycles/time unit). The time invariant lowpass filter used in both anti-aliasing filters has a cutoff frequency of 1/8 cycles/time unit. Thus in the case of uniform sampling the anti-aliasing filter should pass through the first FLOOR(100/8)=12 harmonics of the square wave fundamental frequency.

Figure 1 shows the spectrum of the square wave after being passed through the uniform anti-aliasing time varying filter. Note that only odd harmonics are present, and that the harmonics are passed through up to the 11th harmonic. Harmonics 13 and above are blocked by the anti-aliasing filter. Figure 2 shows the signal after it passes through the uniform anti-aliasing filter. Note the ringing due to the sharp cutoff of the lowpass filter. Figure 3 shows the spectrum of

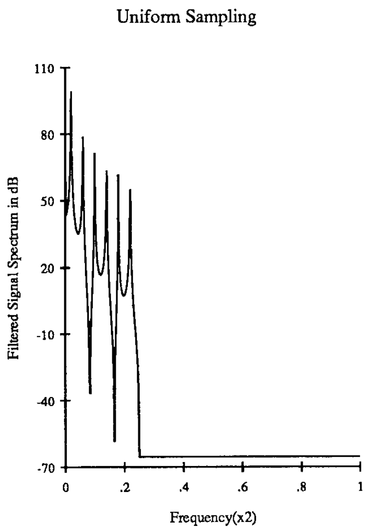

Figure 1: The spectrum of the square wave after passing through the uniform anti-aliasing filter.

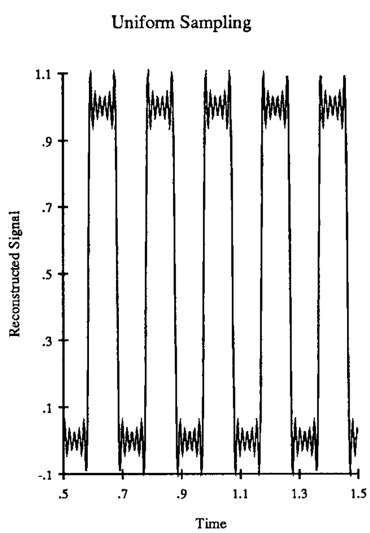

Figure 2: The square wave signal after passing through the uniform anti-aliasing filter.

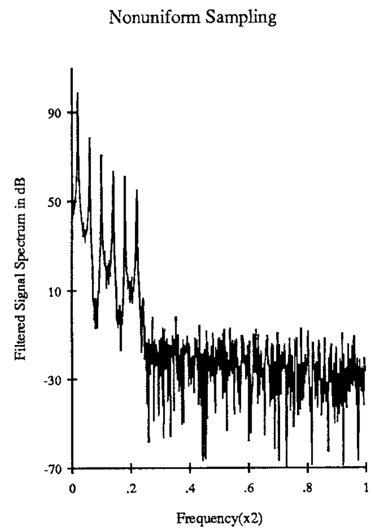

Figure 3: The spectrum of the square wave after passing through the nonuniform anti-aliasing filter.

the square wave after it passes through the nonuniform anti-aliasing filter. Note the significant content in frequencies above the Nyquist frequency. Note also the presence of the higher harmonics of the square wave (those above 11). It is observed that, above a frequency of about 0.4, there seems to be even harmonics present. This harmonic distortion may be due to the above mentioned phase and magnitude distortion by the reconstruction process and it may be that the some of the higher odd harmonics have been shifted in frequency so that they appear to be even harmonics. Another factor that is immediately obvious is the fact that the power in the harmonics in the stop band (the frequencies above 0.125) are about 50 dB below the level they are in the original signal. In fact, if one looks at the antialiased signal, it looks almost identical with the uniform antialiased signal (for this reason we do not show it). One should also notice that the harmonics in the stop band are about 10 dB above the noise. In an attempt to raise the stop band harmonics to a reasonable power level, we applied a stop band "inverse" filter, which amplified the stop band frequencies between the frequencies of 0.125 and 0.22 by 40dB. The frequencies above 0.22 were not amplified (they were actually attenuated by our inverse filter) as the harmonic dis-

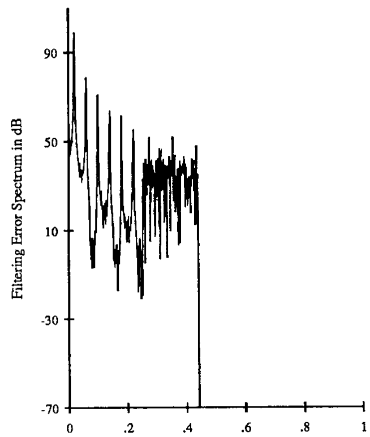

Figure 4: The spectrum of the square wave after passing through the nouniform anti-aliasing filter and the "inverse" filter.

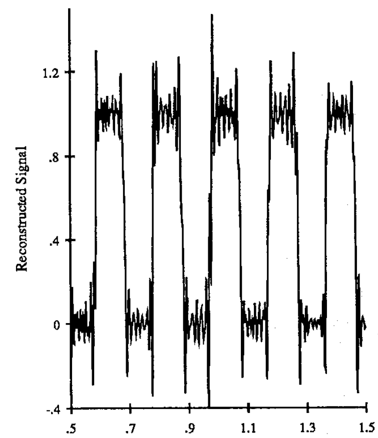

Figure 5: The square wave signal after passing through the nonuniform anti-aliasing filter.

tortion seemed too large in this region. The spectrum of the signal after this inverse filtering step is shown in figure 4. Note that harmonics 13,15,17,19 and 21 have been restored, although their relative levels have been distorted somewhat from what they should be. In figure 5 we show the form of the signal as it leaves the inverse filtering stage. It can be seen that the additional harmonics have sharpened up the square wave edges, but have contributed overshoot of the edges, as well as added some uncorrelated noise to the signal. The effect of the extra harmonics can also be seen in the flat parts of the square wave.

5 Discussion

The work presented here is a preliminary investigation into the use of nonuniform sampling systems for digital storage and transmission of data. There are many questions that needed to be addressed. A few of these are: What is the perceptual effect of the nonuniform sampling/reconstruction techniques on speech and music signals? Is the result actually better in terms of human judgement than uniform sampling/reconstruction methods? How much disorder is needed in the sample sequence to optimize the trade off between the desired whitening or decorrelation of the filtering error and the noise level? Can one develop generic random sample sequences which minimize the noise levels while maximizing the decorrelation? Clearly, if we know something about the class of signals we are dealing with we can to some extent match the sampling sequences to the signal class. For example, in uniform sampling schemes one determines the bandwidth of the signal class and chooses the sampling rate accordingly. For a class of Generalized Phase Modulated signals having a common modulator, one can similarly define a suitable nonuniform sampling sequence. One could also consider systems in which the signal class was estimated from the signal itself and perform an implicit sampling operation. For the case of nonuniform sampling this estimation process is equivalent to a demodulation process [2].

There is a similarity between the filtering error incurred by a sampling/reconstruction process that uses anti-aliasing, and the aliasing error that is incurred by processes that do not use any anti-aliasing. In both cases, uniform sampling results in a reconstruction error that is correlated with the signal in a way that is objectionable to human listeners. In addition to whitening of the error due to anti-aliasing, nonuniform sampling can also whiten the error due to lack of anti-aliasing. Yellot [5] has demonstrated the effectiveness of nonuniform sampling in whitening aliasing error in the human visual system. In the retina there is no anti-aliasing filter and signals with significant high spatial frequency content can be imaged on the photoreceptor array. Yellot has shown that the randomness or disorder of the photoreceptor distribution in the retina breaks down the correlated nature of the aliasing error, reducing the perceptual confusion that results from signal dependent aliasing error.

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