Time-Warped Bandlimited Signals: Sampling, Bandlimitedness, and Uniqueness of Representation

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1. INTRODUCTION

The ability to reconstruct a complex-valued signal on $\mathbb{R}$ from a sequence of sample values $\{f(t_n)\} \subseteq \mathbb{C}$ is desirable in a variety of engineering applications. While this problem is ill-posed in general, many reconstruction formulas of the form

$$f(t) = \sum_{n=\infty}^{\infty} f(t_n) \gamma_n(t)$$

have been obtained for various restricted classes of functions.

It was observed in [1] that such a formula for reconstruction of functions from a given class $\mathcal{C}$ extends directly to a reconstruction formula for functions formed by composition of any $f \in \mathcal{C}$ with an invertible function $\gamma : \mathbb{R} \to \mathbb{R}$. Application of a coordinate transformation such as $\gamma$ to the domain of a signal is commonly called “time-warping” in signal processing literature. Consequently, signals of this type have become known as “time-warped” signals.

Among the most important forms of the type (1) are connected with reconstruction of bandlimited signals; i.e., functions having the form

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} f(\omega) e^{i\omega t} \, d\omega$$

where $f \in L^2(\mathbb{R})$ and $0 < \Omega < \infty$. Motivated by their reconstructability from samples, this note presents some comments on the class $\mathcal{B} \circ \Gamma$ of time-warped bandlimited signals; i.e., functions of the form $f \circ \gamma$ with $f$ belonging to the class $\mathcal{B}$ of bandlimited signals and $\gamma : \mathbb{R} \to \mathbb{R}$ belonging to a class $\Gamma$ of continuous and invertible warping functions.

2. RESULTS

The perspective of Paley and Wiener [3] is natural to consider bandlimited functions on the complex domain is adopted in what follows. It thus becomes necessary to consider warping functions on $\mathbb{C}$ as well. Given a bandlimited function $f : \mathbb{R} \to \mathbb{C}$, denote by $F$ the corresponding entire function with values defined by

$$F(z) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} f(\omega) e^{i\omega z} \, d\omega$$

Similarly, given $h \in \mathcal{B}$, denote by $H$ the associated entire function. Define $G$ to be the collection of all continuous functions $G : \mathbb{C} \to \mathbb{C}$ with restrictions $\gamma$ to $\mathbb{R}$ that are real-valued and bijective. If $G \in \mathcal{G}$ then the corresponding $\gamma \in \Gamma$ is well defined. Thus, given bandlimited functions $F$ and $H$ on the complex domain, finding a $G \in \mathcal{G}$ such that $H = F \circ G$ ensures that there is some $\gamma \in \Gamma$ such that $h = f \circ \gamma$. Given $\gamma \in \Gamma$ such that $h = f \circ \gamma$, however, there is no a priori guarantee that any $G \in \mathcal{G}$ exists with the property that $H = F \circ G$. In this sense, considering complex warping functions in $\mathcal{G}$ is more restrictive than considering real-valued warping functions in $\Gamma$.

Theorem 1: If $f \in \mathcal{B}$ is not identically zero and $G \in \mathcal{G}$, then $H = F \circ G$ is bandlimited if and only if $G$ is affine.

It is clear that $H = F \circ G$ will be bandlimited if $G$ is affine. The proof of the “only if” part of this theorem is based on the growth properties of the entire functions $F$ and $H$. Specifically, it relies on the following results.

Lemma: Suppose $G \in \mathcal{G}$, $f \in \mathcal{B}$ is not identically zero, and $H = F \circ G$ is bandlimited, then $G$ is entire.

Theorem 2 (from [4]): If $F$ and $G$ are entire and the order of $F \circ G$ is finite, then either (1) $G$ is a polynomial and the order of $F$ is finite, or (2) $G$ is a non-polynomial function of finite order and the order of $F \circ G$ is zero.

Theorem 3 (based on results from [4]): If $f \in \mathcal{B}$ is not identically zero and $G$ is a polynomial of degree $n > 1$, then the order of $H = F \circ G$ is greater than one.

The proof of Theorem 1 proceeds as follows. Assuming $H$ is bandlimited, Theorem 1 establishes $G$ is entire. Theorem 2 may be applied to show that $G$ is a polynomial. Theorem 3 implies that the degree of $G$ is either zero or one. If $G$ were constant then $H$ would be constant. Since $h \in L^2[0, \pi]$, it cannot be constant without being identically zero. Thus $G$ is a polynomial of degree exactly one; i.e., $G(z) = az + b$ with $a \neq 0$. The condition that $\gamma$ is real valued implies that $a$ and $b$ are real. Hence $\gamma(z) = az + b$ for real numbers $a$ and $b$ with $a \neq 0$.

3. DEMODULATION

Earlier work [2] has established that $\mathcal{B} \circ \Gamma$ contains all bandlimited functions and many non-bandlimited functions, but not all of $L^2$. A remaining issue is that of demodulation: given $h \in \mathcal{B} \circ \Gamma$, can it be decomposed into a bandlimited function $f$ and a bijective monotone time warping function $\gamma$?

If $h \in \mathcal{B} \circ G$, then there are necessarily many ways to express $h$ as a composition $f \circ \gamma$. Given any $a > 0$, for example, define functions $f_1$ and $\gamma_1$ by $f_1(t) = f(at)$ and $\gamma_1(t) = \gamma(t/a)$. Then $f_1 \in \mathcal{B}$, $\gamma_1 \in \mathcal{G}$, and $f_1 \circ \gamma_1 = f \circ \gamma$. This kind of representational ambiguity can be circumvented by stipulating that $f$ has exactly unit bandwidth. In this case, the question of representational ambiguity may be addressed by a corollary to Theorem 1.

Corollary [of Theorem 1]: Suppose $h = f_1 \circ \gamma_1 = f_2 \circ \gamma_2$ with $f_1$ and $f_2$ having exactly unit bandwidth and $\gamma_1, \gamma_2 \in \mathcal{G}$. Then $f_1(t) = f_2(t-b)$ and $\gamma_1(t) = \gamma_2(t) + b$ for some real constant $b$ and all $t \in \mathbb{R}$.

References


