Tikhonov Stabilizers

Example of a stabilizing functional

Let F be the space of continuous functions on $[a, b]$ with the metric

$$\rho_F(z_1, z_2) = \sup_{a \leq x \leq b} |z_1(x) - z_2(x)|$$

Let F_1 (the subspace over which Ω is defined) be the *Sobolev* space W_2^p, which is the space of functions that have square integrable derivatives up to p^{th} order.

Then, the following can be shown to be a stabilizer defined on F_1:

$$\Omega[z] = \int_a^b \sum_{r=0}^{p} q_r(x) \left(\frac{d^r z}{dx^r}\right)^2 dx$$

where $q_r(x) \geq 0$ for $0 \leq r < p$ and $q_p(x) > 0$.

If $q_r(x) = q_r$ (i.e. constant) this type of functional is called a *Tikhonov Stabilizer*. These stabilizers are the sort that are most commonly used in vision algorithms.

Problem Information Needed in Regularization

In order to solve the inverse problem using regularization we must know the following:

- The data space U and the solution space F
- The metrics for the data and solution spaces
- The operator A
- The stabilizer Ω
- The regularization parameter α
- The data value u

Depending on the nature of the problem, some or all of these items may be not known exactly, and so approximations or assumptions must be made.

In general, the choice of these assumptions is not unique. Thus, there can be more that one approximate solution provided by the regularization approach.

This non-uniqueness might not be a problem, if we don't care about the particular nature of a solution, only that it be an approximation to the "true" solution.

In most cases, it DOES matter what the solution is like. If we make poor approximations or assumptions, the regularization process will still produce a solution, but that solution need not be close to the "true" solution.

Problem Modeling

The process of obtaining an approximation for the operator A is that of $modeling$ the image formation process.

Much of computer vision research is concerned with this modeling process.

The process of determining a suitable stabilizing functional is that of modeling the characteristics of the expected solutions or more simply, that of modeling the scene.

To see that this is the role of the stabilizing functional, let us look at the $Bayesian$ interpretation of regularization.

Bayesian Approach to Regularization

In the Bayesian approach we determine the solution z_α which maximizes the conditional probability $P(z_\alpha|u_\delta)$.

This conditional probabilty can be expressed using Bayes rule as follows:

$$P(z_\alpha|u_\delta) = \frac{P(u_\delta|z_\alpha)P(z_\alpha)}{P(u_\delta)}$$

$P(u|z)$ is the conditional probability of the data u given the solution z. This probability represents our knowledge of the image formation process. It also models the process by which uncertainty is introduced into the data.

Example:

$$P(u|z) = \frac{1}{\delta\sqrt{2\pi}} e^{-(Az-u)^2/2\delta^2}$$

$P(u)$ is the ($a\ priori$) probability density of the data before the measurement is made.

$P(z)$ is the ($a\ priori$) probability density of the solution before the measurement is made. This probability represents our prior knowledge of the solution. It specifies which solutions are more probable than others.

Relationship Between Variational and Bayesian Approaches to Regularization

The variational approach to regularization involves minimizing a functional

$$M^{\alpha}(z, u) = \rho_F(Az, u) + \alpha\Omega[z]$$

The Bayesian approach involves maximizing a conditional probability

$$P(z|u) = \frac{P(u|z)P(z)}{P(u)}$$

These two approaches are equivalent if the $P(z|u)$ is in the form of a "Gibbs" distribution:

$$P(z|u) = \frac{e^{-\beta E(z,u)}}{Z}$$

where Z is a normalization term (sometimes called the partition function).

In a physical interpretation $E(z,u)$ is the free energy of a system corresponding to a configuration z, and $P(z|u)$ is the probability of the system configuration z, given certain measurements u. The most likely system configuration is the one that minimizes the free energy of the system.

We can associate $M^{\alpha}(z,u)$ with the $E(z,u)$ of the Gibbs distribution. Thus we get:

$$P(z|u) = \frac{e^{-M^{\beta}\alpha(z,u)}}{Z} = \frac{e^{-\beta\rho_F(Az,u)}e^{-\beta\alpha\Omega[z]}}{Z}$$

We can then make the following associations

$$P(u|z) = \frac{e^{-\beta\rho_F(Az,u)}}{Z_1}$$

$$P(z) = \frac{e^{-\beta\alpha\Omega[z]}}{Z_2}$$

$$P(u) = \frac{Z}{Z_1 Z_2}$$

It can be seen that A and ρ_F show up in the $P(u|z)$ term, and thus should be selected to reflect our knowledge of the image formation process.

Likewise $\Omega[z]$ shows up in the $P(z)$ term and should therefore reflect our *a priori* knowledge of the solution.

Example of the Use of Regularization in Vision

Optical Flow Computation

Optical Flow = Velocity field on the image plane
2-D projection of 3-D motion

The importance of optical flow to (animal) vision was first noted by J.J. Gibson.

Optical flow is usually the information on which 3-D motion computation is performed.

But the process of Optical Flow computation is itself ill-posed!

Optical flow algorithms often begin with making an assumption:

CONVECTED IRRADIANCE CONSTRAINT:

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

where $\vec{v} = (u, v)$ is the optical flow vector and $I(x, y, t)$ is the image intensity.

If I is smooth in x, y, t we can take derivatives and hence expand I in the LHS of the above equation in a Taylor's series

$$I(x, y, t) + \delta x \frac{\partial I}{\partial x} + \delta y \frac{\partial I}{\partial y} + \delta t \frac{\partial I}{\partial t} + \mathcal{O}(\delta x^2, \delta y^2, \delta t^2) = I(x, y, t)$$

$$u \frac{\partial I(x, y, t)}{\partial x} + v \frac{\partial I(x, y, t)}{\partial y} + \frac{\partial I(x, y, t)}{\partial t} = 0$$

$$\nabla I \cdot \vec{v} = -\frac{\partial I}{\partial t}$$

The convected intensity constraint is not sufficient to provide a unique solution. The resulting ambiguity is known as the *aperture problem*.

Note that $\nabla I \cdot \vec{v} / |\nabla I| = \partial \vec{v} / \partial \hat{s} = -(\partial I / \partial t) / |\nabla I|$, where $\hat{s}$ is the gradient direction.

Therefore we know the component of $\vec{v}$ in the direction of the image brightness gradient, but do not know anything about the component perpendicular to this direction.

We can use regularization to resolve the ambiguity.

1) Horn-Schunk method. Minimize:

$$\int (I_x u + I_y v + I_t)^2 + \lambda [(u_x^2 + u_y^2) + (v_x^2 + v_y^2) dx dy$$

Note the peculiar form of the norm that is used, and the use of a Tikhonov stabilizer (smoothness constraint).

This approach finds the smoothest optic flow field that is most consistent with the convected intensity constraint.

Minimization problems such as in the Horn-Schunk technique can be solved using the calculus of variations. This involves converting the minimization problem to one of solving a (set of) differential equations (the Euler equations).

The Euler equations for the Horn-Schunk problem are (in vector notation):

$$\nabla^2 \vec{v} = (\lambda \nabla I \cdot \vec{v} + I_t) \nabla I$$

Note that whenever $\nabla I = (0, 0)$, i.e. where the image instensity is constant, the Euler equations reduce to

$$\nabla^2 \vec{v} = (0, 0)$$

Thus, the Horn-Schunk method "fills-in" uniform regions with a harmonic flow field. This is the effect of the smoothness constraint, and may not be what is desired!

2) Hildreth's method. Minimize:

$$\int_C \left([\vec{v}(s) \cdot \hat{n}(s) - N(s)]^2 + \lambda \left(\frac{\partial \vec{v}(s)}{\partial s} \right)^2 \right) ds$$

where s is an arc length parameter along a contour C. The contour C is taken to be an image edge segment. $\hat{n}(s)$ is the vector normal to the edge segment and $N(s)$ is the "Normal flow" (the component of the optical flow perpendicular to the edge segment).

This approach does no filling in, as it computes the optical flow only along the edge segments.

The convected intensity constraint is not used, but rather the simpler constraint that the perpendicular component of the computed optical flow should be close to the measured normal flow.

This technique fails (i.e. produces illusions) in ways similar to the human motion perception system.