Visual Motor Integration

VISION $\rightarrow$ ACTION

... visual servoing, tracking, ...

VISION $\leftarrow$ ACTION

... active vision ...

Ill-Posed Problems

see **Solutions of Ill-Posed Problems**, by A.N. Tikhonov and V.Y. Arsenin, V.H. Winston, Washington, 1977

What is meant by the "solution" to a problem?

For example, suppose that we want to find the "solution" z to some problem given "data" u.

Typically this is done by applying a solution operator to the data:

$$z = Ru$$

To define the problem more completely specify that u, z are elements of metric spaces U, F, respectively, with metrics $\rho_U(u_1, u_2)$ and $\rho_F(z_1, z_2)$.

STABILITY: The problem of determining $z = Ru$ in F from $u \in U$ is *stable on the spaces (F,U)* if, for every positve number ϵ the exists a positive number $\delta(\epsilon)$ such that

$$\rho_U(u_1, u_2) < \delta(\epsilon)$$

implies

$$\rho_F(z_1, z_2) \leq \epsilon$$

where $z_1 = Ru_1$ and $z_2 = Ru_2$

Well-Posed Problems

The problem $z = Ru$ is said to be *Well-Posed (in the sense of Hadamard)* on the pair of metric spaces (F, U) if:

- 1) For every element $u \in U$ there exists a solution $z \in F$
- 2) The solution is unique
- 3) The problem is stable on (F, U)

Problems that do not satisfy these conditions are said to be *ill-posed*.

Note that ill-posedness depends on the choice of metric spaces.

Example of an Ill-Posed Problem

Fredholm Integral Equations of the 1st Kind:

$$\int_a^b K(x,s)z(s)ds = u(x) \quad c \leq x \leq d$$

$K(x,s)$ is known, and assumed to be continuous with continuous first derivatives. u is the "data", from which we want to recover z

Note that, if $K(x,s) = K(x-s)$, and $(a,b) = (-\infty,\infty)$ then the Fredholm integral becomes a convolution integral and the problem becomes a deconvolution problem.

We need to specify the metric spaces. Choose F to be the class of functions continuous on $[a,b]$, and U to be the class of functions that have bounded energy on the interval $[c,d]$. Choose the following metrics:

$$\rho_U(u_1,u_2) = [\int_c^d (u_1(x) - u_2(x))^2 dx]^{1/2}$$

$$\rho_F(z_1,z_2) = \max_{s\in[a,b]} |z_1(s) - z_2(s)|$$

Suppose that for some $u = u_1(x)$ the function z_1 is a solution:

$$\int_a^b K(x,s) z_1(s) ds = u_1(x)$$

Now suppose that we can't actually measure $u_1(x)$ exactly, but only an approximate or noisy version of $u_1(x)$, i.e. we know $u_2(x)$ where

$$\rho_u(u_2, u_1) < \delta$$

We could hope to find, then, an approximate solution

$$z_2 = Ru_2$$

which is *close* to z_1, i.e. $\rho_F(z_2, z_1) < \epsilon$.

But is this always possible? The answer is yes, if the problem is well-posed, due to the stability of well-posed problems.

This is not the case, however, for the problem being considered here!

Instability in the Fredholm Inversion

Suppose that our "noisy" data is given by

$$u_2(x) = u_1(x) + N \int_a^b K(x,s)\sin(\omega s)ds$$

It can be shown that the solution to this data is given by

$$z_2(s) = z_1(s) + N\sin(\omega s)$$

The metrics are given by

$$\rho_U(u_2,u_1) = |N|\{\int_c^d\{\int_a^b K(x,s)\sin(\omega s)ds\}dx\}^{1/2}$$

$$\rho_F(z_2,z_1) = \max_{s\in[a,b]} |N\sin(\omega s)| = |N|$$

As $\omega \to \infty$ $\rho_U(u_2,u_1) \to 0$ for all N while $\rho_F(z_2,z_1) = N$.

Thus, this problem violates condition 3 for well-posedness.

It can be shown that the problem can violate the other 2 conditions for well-posedness as well!

Consider the case where u_2 is discontinuous. Such functions exist in U, and so the problem is well-defined. But there is no way for the Fredholm equation to produce a discontinous u. Thus there is no solution z that can produce the data u_2.

This violates condition 1 for well-posedness.

Suppose $K(x, s) = 0$ in some interval $(e, f) \in [a, b]$. Then $z(s)$ can be anything on (e, f) (as long as the resulting $z(s)$ is still in F) and still generate the same data $u(x)$.

This violates condition 2 for well-posedness.

The Fredholm equation problem is an example of an *inverse problem*, which are usually ill-posed.

Almost **all** vision problems are inverse problems and are usually ill-posed!

Vision problems usually violate more than one of the conditions for well-posedness.

Solution of Ill-Posed Problems

Question: How can we "solve" ill-posed problems?

Answer: We can't! It's impossible, or meaningless to do so.

Question: How does the human vision system solve this "impossible" task?

Answer: We're not quite sure...

Question: What can we do?

Answer: We can solve a related problem that *is* well-posed, and that will provide an *approximate solution*.

Tikhonov Regularization

Let us take as a generic inverse problem that of determining $z \in F$ given $u \in U$, where $u = Az$. A is an operator that maps F into U, and whose inverse A^{-1} is assumed to exist. A need not be continuous and the set F need not be compact.

Suppose u_T is the "true" or noise free data generated by the "true" solution z_T.

Let u_δ be such that $\rho_U(u_\delta, u_T) \leq \delta$.

In general, the approximate solution z_δ can not be obtained merely by

$$z_\delta = A^{-1} u_\delta$$

even though A^{-1} exists, as the ill-posedness of the problem causes problems.

Instead we use another operator, call it R which has parameter(s) that depend on the value of δ, which will provide a stable approximate solution.

$$z_\delta = R(u_\delta, \delta)$$

As $\delta \to 0$ (or as $\rho_U(u_\delta, u_T) \to 0$) we must have that $\rho_F(z_\delta, z_T) \to 0$ when $Az_T = u_T$.

Such an operator is called a $Regularizing\ Operator$.

Definition (from Tikhonov and Arsenin):

An operator $R(u, \alpha)$, depdending on a parameter α is called a regularizing operator for the equation

$$Az = u$$

in a neighborhood of $u = u_T$ if:

1) There exists a positive number δ_1 such that the operator $R(u, \alpha)$ is defined for exery $\alpha > 0$ and every $u \in U$ for which $\rho_U(u, u_T) \leq \delta < \delta_1$ and

2) there exists a function $\alpha = \alpha(\delta)$ of δ such that for every $\epsilon > 0$ there exists a number $\delta(\epsilon) \leq \delta$ such that the inclusion $u_\delta \in U$ and the inequality

$$\rho_U(u_T, u_\delta) \leq \delta(\epsilon)$$

imply

$$\rho_F(z_T, z_\alpha) \leq \epsilon$$

where $z_\alpha = R(u_\delta, \alpha(\delta))$

The second condition in the definition says that as long as $u_\delta \in U$ (i.e. that u is a valid data point) and u_δ is "close" to u_T then the solution z_α provided by the regularizing operator is "close" to the true solution z_T.

z_α is called the *Regularized Solution* of $Az = u$.

The parameter α is called the *Regularization Parameter*.

It is required that α be consistent with the accuracy δ of the data u_δ.

The task of finding an approximate solution of $Az = u$ that is stable under small changes in the data reduces to:

- a) Finding a regularizing operator.
- b) Determining the value of the regularizing parameter α.

Construction of Regularizing Operators

Variational Approach:

Definition:

Let $\Omega[z]$ be a continuous non-negative functional defined on a subset F_1 of F that is everywhere dense in F, and that

a) z_T belongs to the domain of definition of $\Omega[z]$ and
b) for every positive number d the set of elements $z \in F_1$ for which $\Omega[z] \leq d$ is a compact subset of F_1.

Such Ω's are called Stabilizing Functionals or Stabilizers.

Let Q_δ be the set defined by all values z such that

$$\rho_U(Az, u_\delta) \leq \delta$$

that is, all values which produce data that are within a distance δ of u_δ.

We could obtain an approximate solution by just taking any element of Q_δ, call it z_δ. But z_δ will, in general, not be continuous wrt δ. And it may be difficult to determine Q_δ.

Instead, use the following variational principle:

Let $\Omega[z]$ be a $stabilizer$ defined on $F_1 \in F$. Consider only those elements of Q_δ for which $\Omega[z]$ is defined, i.e. $Q_\delta \cap F_1 = F_{1,\delta}$.

Among the elements of this set, find the one (or ones) which $minimize$ the functional $\Omega[z]$

Let z_δ denote such an element (which can be shown to always exist).

z_δ can be thought of as resulting from applying an operator R, that depends on δ to u_δ. It can be shown that this operator is a regularizing operator!

Determining the subset $F_{1,\delta}$ is often difficult, due to the difficulty in finding Q_δ.

Another approach can be used:

Define $\Omega_0 = \inf_{z \in F_1} \Omega[z]$ and let M_0 denote the set of all elements z of F_1 such that $\Omega[z] = \Omega_0$.

Assume for simplicity that M_0 consists of a single element, z_0 (the multiple solution case can be handled as well).

There are two possible cases: either M_0 lies in $F_{1,\delta}$ or it doesn't.

In the first case, we can just take z_0 as our solution and we are done.

In the second case $\rho_U(u_\delta, AM_0) \geq \delta$ (by definition of $F_{1,\delta}$). But it can be shown that the greatest lower bound of $\Omega[z]$ on $F_{1,\delta}$ is attained for z_δ where $\rho_U(Az_\delta, u_\delta) = \delta$.

Thus we can solve the following constrained optimization problem instead:

Take as the solution the element $z \in F_1$ that minimizes $\Omega[z]$ subject to the constraint that

$$\rho_U(Az, u_\delta) = \delta$$

This is a constrained optimization problem that can be solved using Lagrange Multipliers, that is, by minimizing the functional:

$$M^\alpha[z, u_\delta] = \rho_U(Az, u_\delta) + \alpha\Omega[z]$$

where the Lagrange multiplier, α, is determined from the condition $\rho_U(Az_\alpha, u_\delta) = \delta$, where z_α is a solution $z \in F_1$ at which $M^\alpha[z, u_\delta]$ is at a minimum.

Note that α depends on δ as well as A and u_δ.

$$z_\alpha = R_1(u_\delta, \alpha), \qquad \alpha = \alpha(\delta)$$

R_1 is a regularizing operator and z_α is an approximate solution to $Az = u$ for $\rho_U(u, u_T) \leq \delta$.

This approach to regularization has found wide application in computer vision.