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Necessary Conditions for Consistency of Noise-Free, Closed-Loop Frequency-Response Data with Coprime Factor Models

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Abstract

A basic requirement of robust control theory is that a nominal model and an uncertainty model be available for the plant. The assumption is that the plant can be modeled by at least one of the perturbations of the nominal model in the uncertainty set. This raises the problem of constructing such an uncertainty set that would be consistent with a given set of experimental input-output data. This paper introduces necessary conditions for the model/data consistency problem with coprime factor uncertainty and noise-free closed-loop frequency-response measurements. It is assumed that a stabilizing controller providing sufficient damping was implemented on a lightly-damped or unstable system, allowing measurement of the closed-loop frequency response at distinct frequencies. The necessary conditions involve the computation of singular values of complex matrices associated with the measurement frequencies. Standard factorizations and left-coprime factor models of large flexible space structures are considered.

1 Introduction

The closed-loop model/data consistency problem for a family of coprime factorizations is more difficult than the open-loop one, but potentially very useful. Many systems are very lightly damped

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or unstable, and perturbed coprime factorizations are often a good choice to model them [6]. It may be difficult or even impossible to run open-loop frequency-response experiments on these systems, so the open-loop results for model/data consistency given in [2], [4] and [5] may be of limited use. In this paper, it is assumed that a stabilizing controller providing sufficient damping was implemented on such a system, allowing measurement of the closed-loop frequency response at distinct frequencies. We will see that these measurements can be used to refine the norm bound $|\boldsymbol{r}(j\omega)|$ on the coprime factor uncertainty. This improved characterization of the uncertainty in the model allows the design of a better controller achieving desired robust performance goals.

It should be noted that finding a necessary and sufficient condition for consistency of the model with closed-loop data is more difficult than in the open-loop case. The reason is that we must not only show the existence of a stable perturbation of ∞-norm less than one interpolating a set of complex matrices, but we must also show that there exists such a perturbation that stabilizes the nominal closed-loop system. That is, we must find a strongly stabilizing perturbation in the uncertainty set interpolating the frequency-response data. Thus, we will only give necessary conditions for the problems formulated.

Assume that we have N noise-free frequency-response data points measured at N distinct frequencies. A necessary condition for the noise-free coprime-factor model/data consistency problem is given as a simple test consisting of computing minimum-norm solutions to N underdetermined linear complex matrix equations, just like the open-loop case in [2]. For the case of a special factorization for a square $p \times p$ large flexible space structure (LFSS) introduced in [3], a necessary condition based on the Schmidt-Mirsky Theorem is derived. It requires the computation of the p^{th} singular values of N complex linear fractional transformations (LFT). The theorem on boundary interpolation in $\mathcal{RH}_\infty$ in [2] is used in those two cases. A numerical example using a left-coprime factorization (LCF) of Daisy, an LFSS experimental testbed, is worked out to illustrate the results.

Notation

Let H be an $n \times m$ complex matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_q$, $q := \min\{m, n\}$. The maximum and minimum singular values of H are written as $\overline{\sigma}(H) = \sigma_1$ and $\underline{\sigma}(H) = \sigma_q$ respectively. The norm of H is taken to be its maximum singular value: $\|H\| = \overline{\sigma}(H)$, and H^* is its conjugate transpose. The space $\mathcal{H}_\infty$ is the class of functions analytic in the open right half-plane and bounded on the imaginary axis with norm defined as $\|Q\|_\infty = \sup_{\omega \in \mathbb{R}} \|Q(j\omega)\|$. The prefix $\mathcal{R}$ on $\mathcal{RH}_\infty$ denotes real-rational; so $\mathcal{RH}_\infty$ is the class of scalar or matrix-valued proper stable real-rational transfer functions. According to the context, it should be clear whether we are considering scalar or matrix-valued functions, but sometimes we will write, say, $\mathcal{H}_\infty^{m\times n}$. For a normed space $\mathcal{X}$, $\mathcal{BX}$ denotes its open unit ball. A function in $\mathcal{H}_\infty$ ($\mathcal{RH}_\infty$) is a unit if its inverse also belongs to $\mathcal{H}_\infty$ ($\mathcal{RH}_\infty$). Upper and lower linear fractional transformations are now defined. Suppose the matrix P is partitioned as $P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, K is such that its transpose K^T has the same dimensions as P_{22}, and Δ is such that its transpose Δ^T has the same dimensions as P_{11}. Then the lower linear fractional transformation (LFT) of P by K is $\mathcal{F}_L(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$ whenever the inverse exists. The upper LFT of P by Δ is $\mathcal{F}_U(P, \Delta) := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}$ whenever the inverse exists.

2 Problem Statement

Let us consider the model/data consistency problem for a closed-loop finite-dimensional linear time-invariant system. The data set consists of N complex matrices (or complex scalars for SISO systems) obtained by running noise-free closed-loop frequency-response experiments on the closed-loop system at N distinct frequencies. The uncertainty set is composed of norm-bounded factor perturbations in $\mathcal{RH}_\infty$.

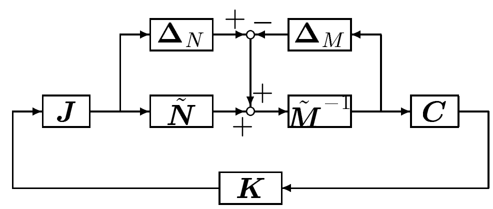

Figure 1: Feedback control of a perturbed LCF model

Let the open-loop nominal plant model $\boldsymbol{G}$ be a proper real-rational transfer matrix. Let the square, invertible $\tilde{\boldsymbol{M}}$ and $\tilde{\boldsymbol{N}}$ in $\mathcal{RH}_\infty$ be left-coprime, $\boldsymbol{C}$ be an output transfer matrix and $\boldsymbol{J}$ be a diagonal input transfer matrix, both in $\mathcal{RH}_\infty$, such that the nominal plant model can be factorized as $\boldsymbol{G} = \boldsymbol{C}\tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}}\boldsymbol{J}$. The matrices $\boldsymbol{C}$ and $\boldsymbol{J}$ are included for compatibility with the special factorization for large flexible space structures introduced in [3]. When set to identity matrices, a standard left-coprime factorization of $\boldsymbol{G}$ is obtained. Let the perturbed open-loop plant model $\boldsymbol{G}_p$ be expressed as a perturbed factorization with $\tilde{\boldsymbol{M}}_p$, $\tilde{\boldsymbol{N}}_p \in \mathcal{RH}_\infty$

$$\boldsymbol{G}_p = \boldsymbol{C}\tilde{\boldsymbol{M}}_p^{-1}\tilde{\boldsymbol{N}}_p\boldsymbol{J}, \tag{1}$$

where $\tilde{\boldsymbol{M}}_p = \tilde{\boldsymbol{M}} + \boldsymbol{\Delta}_M$, $\quad \tilde{\boldsymbol{N}}_p = \tilde{\boldsymbol{N}} + \boldsymbol{\Delta}_N$, $\quad \boldsymbol{\Delta}_M, \boldsymbol{\Delta}_N \in \mathcal{RH}_\infty$. Define the uncertainty matrix $\boldsymbol{\Delta} := [\boldsymbol{\Delta}_N \;\; -\boldsymbol{\Delta}_M]$. Clearly, $\boldsymbol{\Delta} \in \mathcal{RH}_\infty$. Define the uncertainty set

$$\mathcal{D}_r := \{\boldsymbol{\Delta} \in \mathcal{RH}_\infty \;:\; \|\boldsymbol{r}^{-1}\boldsymbol{\Delta}\|_\infty < 1\} \tag{2}$$

and the family of plants

$$\mathcal{P} := \{\boldsymbol{G}_p \;:\; \boldsymbol{\Delta} \in \mathcal{D}_r\}\ , \tag{3}$$

where $\boldsymbol{r}$ is a unit in $\mathcal{RH}_\infty$. The unit $\boldsymbol{r}$ characterizes the size of the uncertainty in the coprime factors at each frequency ω because $\|\boldsymbol{r}^{-1}\boldsymbol{\Delta}\|_\infty < 1$ implies $\|\boldsymbol{\Delta}(j\omega)\| < |\boldsymbol{r}(j\omega)|$. Assuming that $\boldsymbol{K}$ internally stabilizes $\boldsymbol{G}$, we have from [6] the result that the closed-loop system of Figure 1 with controller $\boldsymbol{K}$ is internally stable for every $\boldsymbol{G}_p \in \mathcal{P}$ iff

$\left\| \boldsymbol{r} \begin{bmatrix} \boldsymbol{JKC}(I - \boldsymbol{GJKC})^{-1}\tilde{\boldsymbol{M}}^{-1} \\ (I - \boldsymbol{GJKC})^{-1}\tilde{\boldsymbol{M}}^{-1} \end{bmatrix} \right\|_\infty \leq 1$. This result provides the main motivation to make $\boldsymbol{r}$ as small as possible.

In order to be able to use this robust stability result in the design of a robust controller for a real plant, one has to construct and modify the bound $|\boldsymbol{r}(j\omega)|$ until it properly captures the uncertainty in the physical system. One way to do this is to start with a nominal model and a first approximation for $\boldsymbol{r}$, and then use experimental data to check if $|\boldsymbol{r}(j\omega)|$ is large enough to account for the full data set. The necessary tests proposed in this paper are suitable for that purpose. The generic closed-loop model/data consistency problem considered here can be stated as follows: Given noise-free frequency-response data $\{\Phi_i\}_{i=1}^N$ obtained on the closed-loop system at the distinct frequencies $\omega_1, \ldots, \omega_N$, could the data have been produced by at least one plant model in $\mathcal{P}$? Note that it is assumed throughout that the plant and the controller are linear.

3 Necessary Conditions for Model/Data Consistency

Consider the feedback system in Figure 2. Two controllers were included in order to treat the two different configurations of input tracking ($\boldsymbol{K}_1 = I_p$) and input disturbance rejection ($\boldsymbol{K}_2 = I_m$) in a unified way. The tracking configuration is generally used to ensure that the output of the plant $y(t) \in \mathbb{R}^p$ tracks the reference input $v(t) \in \mathbb{R}^p$ over a given frequency band. On the other hand, the input disturbance rejection configuration is used to attenuate the effect of an input disturbance $v(t) \in \mathbb{R}^m$ on the output of the plant $y(t) \in \mathbb{R}^p$. This configuration may facilitate frequency-response experiments on a mechanical system with a force/torque input v that can be applied by control actuators or external ones.

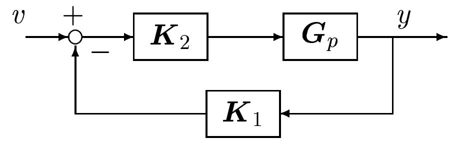

Figure 2: Feedback configuration for input tracking or disturbance rejection.

3.1 Standard Factorization

Here we treat the case of a standard LCF of $\boldsymbol{G}$, i.e., $\boldsymbol{J} = \boldsymbol{C} = I_p$.

Let the transfer matrix from v to y in Figure 2 be denoted as $\boldsymbol{T}$. Then the closed-loop equation is

$$\begin{aligned} \boldsymbol{T} &= (I + \boldsymbol{G}_p\boldsymbol{K}_2\boldsymbol{K}_1)^{-1}\boldsymbol{G}_p\boldsymbol{K}_2 \qquad (4) \\ &= (\tilde{\boldsymbol{M}} + \boldsymbol{\Delta}_M + \tilde{\boldsymbol{N}}\boldsymbol{K}_2\boldsymbol{K}_1 + \boldsymbol{\Delta}_N\boldsymbol{K}_2\boldsymbol{K}_1)^{-1}(\tilde{\boldsymbol{N}} + \boldsymbol{\Delta}_N)\boldsymbol{K}_2 \ . \end{aligned}$$

After rearranging this equation, we get

$$\boldsymbol{\Delta}_N\boldsymbol{K}_2(I - \boldsymbol{K}_1\boldsymbol{T}) - \boldsymbol{\Delta}_M\boldsymbol{T} = \tilde{\boldsymbol{M}}\boldsymbol{T} + \tilde{\boldsymbol{N}}\boldsymbol{K}_2(\boldsymbol{K}_1\boldsymbol{T} - I) \ . \qquad (5)$$

We now state the model/data consistency problem for closed-loop frequency-response data for a feedback control system as in Figure 2.

Problem 1 *We are given noise-free closed-loop frequency-response data* $\{\Phi_i\}_{i=1}^N$ *obtained on the closed-loop system of Figure 2 at the distinct frequencies* $\omega_1, \ldots, \omega_N$*. Could the data have been produced by at least one plant model in* $\mathcal{P}$*? Or, in other words, does there exist a fixed* $\boldsymbol{\Delta} \in \mathcal{D}_r$ *such that the closed-loop transfer matrix* $\boldsymbol{T}$ *in (4) with the corresponding perturbed model* $\boldsymbol{G}_p$ *is stable and interpolates the complex matrices* Φ_i *at* $\omega_1, \ldots, \omega_N$*?*

A necessary condition for this question to have a positive answer can be obtained using exactly the same procedure as in the open-loop case [2]. For a measurement frequency ω, let $U := [\tilde{\boldsymbol{M}}\boldsymbol{T} +$

$\tilde{\boldsymbol{N}}\boldsymbol{K}_2(\boldsymbol{K}_1\boldsymbol{T}-I)](j\omega)$ and $W := \begin{bmatrix} \boldsymbol{K}_2(I-\boldsymbol{K}_1\boldsymbol{T}) \\ \boldsymbol{T} \end{bmatrix}(j\omega)$. Then (5) at frequency ω can be written as

$$\Delta W = U \ , \tag{6}$$

where $W \in \mathbb{C}^{(m+p)\times p}$, $U \in \mathbb{C}^{p\times p}$ and $\Delta \in \mathbb{C}^{p\times(m+p)}$ for the tracking configuration, and $W \in \mathbb{C}^{(m+p)\times m}$, $U \in \mathbb{C}^{p\times m}$ and $\Delta \in \mathbb{C}^{p\times(m+p)}$ for the input disturbance rejection configuration. Equation (6) is an underdetermined system of linear equations over the field $\mathbb{C}$. Let $\Delta_i := \boldsymbol{\Delta}(j\omega_i)$ for $i = 1,\ldots,N$, with similar definitions for W_i and U_i. Note that $U_i^* \subset \mathrm{Ra}\{W_i^*\}$, so there exist an infinity of solutions to (6). If W_i does not have full column rank, then the redundant equations can be deleted from (6). After these equations are removed, the new W_i has full column rank. Then the matrix equation (6) can be solved with $\boldsymbol{T} = \Phi_i$ for a minimum-norm Δ_i, $i = 1,\ldots,N$, for example with

$$\Delta_i = U_i(W_i^* W_i)^{-1} W_i^* \ . \tag{7}$$

The following theorem gives a necessary condition for consistency of the perturbed coprime factor model of the plant with the closed-loop frequency-response data.

Theorem 1 *The noise-free closed-loop model/data consistency problem of Problem 1 has a positive answer only if* $\|\Delta_i\| < |\boldsymbol{r}(j\omega_i)|$ *for all* $i = 1,\ldots,N$.

Proof Problem 1 has a positive answer only if there exists a perturbation $\boldsymbol{\Delta} \in \mathcal{D}_r$ interpolating Δ_i at ω_i, $i = 1,\ldots,N$. As in the proof of Theorem 2 in [2], such a function exists iff $\|\Delta_i\| < |\boldsymbol{r}(j\omega_i)|$ for $i = 1,\ldots,N$. ∎

Just as in the open-loop case, the bound $|\boldsymbol{r}(j\omega)|$ can be adjusted such that the inequality in the theorem statement is satisfied for all i. This is necessary for the new model to be consistent with all the data.

3.2 Special Factorization for Square LFSS Models

We now derive a necessary condition for consistency of a factorization of a square, p-input, p-output LFSS model introduced in [3] with closed-loop frequency-response data. More specifically, the factorization of (1) will be used. We consider the setup of Figure 2 for tracking or input disturbance rejection. Two standing assumptions in this section are the following:

(A3.1) $\left[\tilde{\boldsymbol{N}}_p \;\; -\tilde{\boldsymbol{M}}_p\right](j\omega)$ has full row rank for all $\boldsymbol{\Delta} \in \mathcal{D}_r$ and for all $\omega \in \mathbb{R}$.

(A3.2) For all $\boldsymbol{\Delta} \in \mathcal{D}_r$, no pole-zero cancellation occurs in $\overline{\mathbb{C}}_+$ when the product $C\tilde{\boldsymbol{M}}_p^{-1}$ is formed.

Motivation for Assumption (A3.1) is now discussed. It was found empirically that the minimum distance between $\underline{\sigma}\left\{\left[\tilde{\boldsymbol{N}} \;\; -\tilde{\boldsymbol{M}}\right](j\omega)\right\}$ and $|\boldsymbol{r}(j\omega)|$ across frequency for an LCF of Daisy is a good *a priori* indication of the achievable robustness and performance levels with a controller to be designed. The closer this minimum distance was to zero, the harder it was to achieve the performance specification while maintaining robustness to the uncertainty in the modal parameters. If this distance is greater than zero, then the full row rank assumption above is satisfied.

Another way to state Assumption (A3.2) is that the pair $(C,\ \tilde{\boldsymbol{M}}_p)$ is right-coprime for every $\boldsymbol{\Delta} \in \mathcal{D}_r$. This assumption is quite mild; without it, robust internal stability for all $\boldsymbol{\Delta} \in \mathcal{D}_r$ could not be achieved.

The main result of this section is Theorem 2 which gives a necessary condition for a positive answer to the noise-free closed-loop consistency problem for flexible systems, Problem 3.

Consider the following consistency equation at frequency ω illustrated in Figure 3, where the input transfer matrix $\boldsymbol{J}$ has been absorbed into $\boldsymbol{K}_2$:

$$\Phi - \left[I + C(\tilde{M} + r\tilde{\Delta}_M)^{-1}(\tilde{N} + r\tilde{\Delta}_N)K_2K_1\right]^{-1} C(\tilde{M} + r\tilde{\Delta}_M)^{-1}(\tilde{N} + r\tilde{\Delta}_N)K_2 = 0. \qquad (8)$$

It is assumed that:

(A3.3) $\boldsymbol{K}_2(j\omega)$ is nonsingular for all $\omega \in \mathbb{R}$,

(A3.4) the combination of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ internally stabilizes the plant and its nominal model,

(A3.5) $n \geq p$, i.e., there are more modes in the model than there are inputs (and outputs),

(A3.6) C has full row rank.

Assumptions (A3.5) and (A3.6) hold for most LFSS or experimental testbeds and are not really restrictive. Some motivation for these two assumptions is provided by the following observation. Referring to Figure 3, we can see that a necessary condition for consistency is that the columns of the $p \times p$ matrix $(\Phi - H_{22})$ lie in $\text{Ra}\,\{H_{21}\}$ where H_{21} is $p \times n$. But we have to assume that $\Phi - H_{22}$ is nonsingular for Lemma 1 to hold true, so it follows that H_{21} must have full row rank p. This is turn implies that we must have $n \geq p$.

In LFT notation, Equation (8) takes the simpler form:

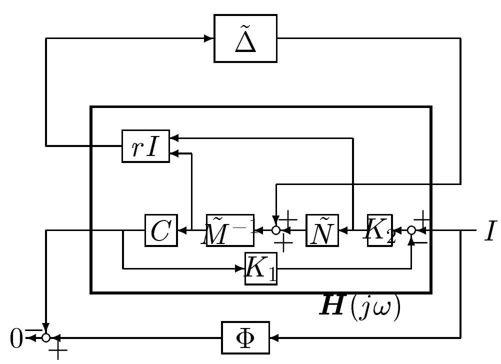

Figure 3: Block diagram of consistency equation for a noise-free feedback-controlled MIMO flexible system.

$$\Phi - \mathcal{F}_U(H, \tilde{\Delta}) = 0, \tag{9}$$

where

$$H := \boldsymbol{H}(j\omega) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

$$H_{11} := \begin{bmatrix} -r(I + K_2K_1C\tilde{M}^{-1}\tilde{N})^{-1}K_2K_1C\tilde{M}^{-1} \\ r(I + \tilde{M}^{-1}\tilde{N}K_2K_1C)^{-1}\tilde{M}^{-1} \end{bmatrix}, \quad H_{12} := \begin{bmatrix} r(I + K_2K_1C\tilde{M}^{-1}\tilde{N})^{-1}K_2 \\ r(I + \tilde{M}^{-1}\tilde{N}K_2K_1C)^{-1}\tilde{M}^{-1}\tilde{N}K_2 \end{bmatrix},$$

$$H_{21} := (I + C\tilde{M}^{-1}\tilde{N}K_2K_1)^{-1}C\tilde{M}^{-1}, \qquad H_{22} := (I + C\tilde{M}^{-1}\tilde{N}K_2K_1)^{-1}C\tilde{M}^{-1}\tilde{N}K_2 \ .$$

This general feedback configuration includes as special cases the reference tracking configuration ($\boldsymbol{K}_1 = I_p$), and the input disturbance rejection configuration ($\boldsymbol{K}_2 = \boldsymbol{J}$). The consistency problem at frequency ω can be stated as follows.

Problem 2 *Given an invertible, noise-free, closed-loop frequency-response datum $\Phi \in \mathbb{C}^{p\times p}$ at ω, does there exist a $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+p)}$ such that $I - H_{11}\tilde{\Delta}$ is nonsingular and $\Phi - \mathcal{F}_U(H, \tilde{\Delta}) = 0$?*

But the more general model/data consistency problem that we want to solve here is the following.

Problem 3 *Given invertible, noise-free, closed-loop frequency-response data $\{\Phi_i\}_{i=1}^N \subset \mathbb{C}^{p\times p}$ at the distinct frequencies $\omega_1, \ldots, \omega_N$, could they have been produced by at least one model in $\mathcal{P}$? Or, in other words, does there exist a $\boldsymbol{\Delta} \in \mathcal{D}_r$ that stabilizes $\boldsymbol{H}$ and such that $\mathcal{F}_U\left[\boldsymbol{H}(j\omega_i), \tilde{\boldsymbol{\Delta}}(j\omega_i)\right] = \Phi_i$, for $i = 1, \ldots, N$?*

We first look for a solution to Problem 2. Recall that K_2 is assumed to have full rank in (A3.3), and hence H_{12} has full column rank. Let $H_{12}^\dagger \in \mathbb{C}^{p\times(p+n)}$ be the Moore-Penrose left-inverse of H_{12}.

It is easy to show that the left-nullspace $\mathcal{N}_L\{H_{12}\} = \text{row span}\left([\tilde{N} \ -\tilde{M}]\right)$. We have the following lemma whose proof closely parallels the proof of Lemma 4 in [2] (see [1] for a complete proof.)

Lemma 1 *For $\omega \in \mathbb{R}_+$, we are given the invertible frequency-response datum $\Phi \in \mathbb{C}^{p\times p}$. Assume $\Phi - H_{22}$ is invertible and $I - H_{11}\tilde{\Delta}$ is nonsingular. Then for $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(p+n)}$, the following consistency conditions are equivalent.*

(a) $\Phi - \mathcal{F}_U(H, \tilde{\Delta}) = 0$

(b) $\left(H_{12}^{\dagger} - Q\begin{bmatrix} \tilde{N} & -\tilde{M} \end{bmatrix}\right)\left[I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}\right] = 0$ *for some* $Q \in \mathbb{C}^{p\times n}$

(c) $\text{rank}\left[I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}\right] \leq n$

This result leads us to the following minimization problem already encountered in [2].

Problem 4 *Compute* $\beta := \inf\left\{\|\tilde{\Delta}\| : \text{rank}\left\{I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}\right\} \leq n,\ \tilde{\Delta} \in \mathbb{C}^{p\times(p+n)}\right\}$.

A solution to this problem is readily given by Theorem 7 of [2]: $\beta = \sigma_p[\mathcal{F}_L(H, \Phi^{-1})]^{-1}$. The only difference with the open-loop case of [2] is that the nonsingularity of $I - P_{11}\tilde{\Delta}$ for all $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+p)}$ was guaranteed by the assumption that $\underline{\sigma}\left[\tilde{\boldsymbol{M}}(j\omega)\right] > |\boldsymbol{r}(j\omega)|,\ \forall\omega$. Here, nonsingularity of $I - H_{11}\tilde{\Delta}$ for all $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+p)}$ is equivalent to robust stability of the closed-loop system of Figure 3 with the constant matrices replaced by their corresponding transfer matrices. This is certainly too strong an assumption. Indeed, if the combination $\boldsymbol{K}_2\boldsymbol{K}_1$ is already a controller providing robust stability, why bother refining the model to design a new robust controller? Instead, we will show in the following lemma (Lemma 2) that if the factor perturbation $\tilde{\Delta}$ renders the matrix $I - \boldsymbol{H}_{11}(j\omega)\tilde{\Delta}$ singular, then $\exists\tilde{\Delta}_0$ as close to $\tilde{\Delta}$ as desired and with the same properties, but that makes $I - \boldsymbol{H}_{11}(j\omega)\tilde{\Delta}_0$ nonsingular. This is the last technical result needed before we can give a solution to Problem 2. The proof is rather long and hence not given here (see [1].)

Lemma 2 *Suppose that for* $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+p)}$, rank$\left\{I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}\right\} = n$ *and* $I - H_{11}\tilde{\Delta}$ *is singular. Then for* $\epsilon > 0$, *there exists a* $\tilde{\Delta}_0$ *with* $\|\tilde{\Delta}_0 - \tilde{\Delta}\| < \epsilon$ *such that* rank$\left\{I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}_0\right\} \leq n$ *and* $I - H_{11}\tilde{\Delta}_0$ *is nonsingular.*

We are now in a position to establish the following result which provides an answer to Problem 2, the noise-free closed-loop MIMO consistency problem for square flexible systems at a single frequency. The proof makes use of the solution to Problem 4 given above.

Lemma 3 *For an invertible noise-free frequency-response datum* $\Phi \in \mathbb{C}^{p\times p}$ *obtained at frequency* ω, *Problem 2 has a positive answer iff* $\sigma_p\left[\mathcal{F}_L(H, \Phi^{-1})\right]^{-1} < 1$.

Proof *Sufficiency* Suppose $\sigma_p\left[\mathcal{F}_L(H, \Phi^{-1})\right]^{-1} < 1$. By Theorem 7 in [2], $\exists\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+p)}$ such that rank$\left\{I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}\right\} = n$. Moreover, Lemma 2 says that $\exists\tilde{\Delta}_0$ as close to $\tilde{\Delta}$ as wanted such that $I - H_{11}\tilde{\Delta}_0$ is nonsingular and rank$\left\{I - \mathcal{F}_L(H, \Phi^{-1})\tilde{\Delta}_0\right\} \leq n$. In particular, we can select it such that $\|\tilde{\Delta}_0 - \tilde{\Delta}\| < 1 - \|\tilde{\Delta}\|$, in which case $\|\tilde{\Delta}_0\| < 1$. Then Lemma 1 says that this perturbation is consistent with the datum, i.e., $\Phi - \mathcal{F}_U(H, \tilde{\Delta}_0) = 0$.
Necessity Suppose $\sigma_p\left[\mathcal{F}_L(H, \Phi^{-1})\right]^{-1} \geq 1$. Then by Lemma 1, there exists no $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+p)}$ such that $\Phi - \mathcal{F}_U(H, \tilde{\Delta}) = 0$ and $I - H_{11}\tilde{\Delta}$ is nonsingular. ■

Finally, a necessary condition is given for the noise-free closed-loop MIMO consistency problem for square flexible systems, Problem 3.

Theorem 2 *The closed-loop model/data consistency problem of Problem 3 has a positive answer only if* $\sigma_p\left\{\mathcal{F}_L\left[\boldsymbol{H}(j\omega_i), \Phi_i^{-1}\right]\right\}^{-1} < 1$ *for all* $i = 1, \ldots, N$.

Proof A necessary condition for Problem 3 to have a positive answer is that there exist a normalized perturbation $\tilde{\boldsymbol{\Delta}} \in \mathcal{BRH}_\infty$ interpolating the minimum-norm $\tilde{\Delta}_i$ satisfying (9) with the datum Φ_i and such that $I - \boldsymbol{H}_{11}(j\omega_i)\tilde{\Delta}_i$ is nonsingular, for $i = 1, \ldots, N$. By the boundary interpolation theorem (Theorem 6) in [2] and Lemma 3, such a function exists iff $\sigma_p \left\{\mathcal{F}_L \left[\boldsymbol{H}(j\omega_i), \Phi_i^{-1}\right]\right\}^{-1} < 1$ for $i = 1, \ldots, N$. ∎

The condition in Theorem 2 is obviously not sufficient as $\boldsymbol{\Delta}$ must also be stabilizing. Again, the bound $|\boldsymbol{r}(j\omega)|$ can be modified such that the inequality in the theorem statement is satisfied for all i. This is necessary to make the new model consistent with all the data.

3.2.1 Numerical Example

The plant model is for Daisy. The perturbed coprime factor model of Daisy developed in [3] for the collocated $\mathcal{H}_\infty$ controller design will be used. The nominal factorization has the form $\boldsymbol{G} = C\tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}}\boldsymbol{J}$ where $\tilde{\boldsymbol{M}},\ \tilde{\boldsymbol{N}} \in \mathcal{RH}_\infty^{23\times 23}$, $\boldsymbol{J} \in \mathcal{RH}_\infty^{23\times 23}$ is diagonal, $C \in \mathbb{R}^{23\times 23}$. In terms of the input and output matrices of [3], $\boldsymbol{J} = \gamma J_2 \boldsymbol{T}_a$ and $C = d_{max}^{-1} C_1 J_1$, where $\boldsymbol{T}_a$ is a diagonal transfer matrix modeling actuator dynamics.

The unit $\boldsymbol{r}$ bounding the factor uncertainty is $\boldsymbol{r}(s) = \frac{0.001s+1.414}{2.32s+1}$. This $\boldsymbol{r}$ makes $\mathcal{P}$ include all perturbations of $\boldsymbol{G}$ induced by variations in the modal parameters to within 10% uncertainty in the modal frequencies, 50% uncertainty in the modal damping ratios, and 8% uncertainty in the entries of the input matrix B_1. One of the plant models perturbed by variations in the modal parameters within the limits given above was randomly selected to be the actual plant $\boldsymbol{G}_a$, and a simple decentralized 23^{rd}-order controller $\boldsymbol{K}_1$ was designed to stabilize it, as well as the nominal plant model. This controller is composed of 23 first-order lead compensators implementing local feedback loops.

The noise-free frequency-response data $\{\Phi_i\}_{i=1}^{50} \subset \mathbb{C}^{23\times 23}$ computed at 50 distinct frequencies

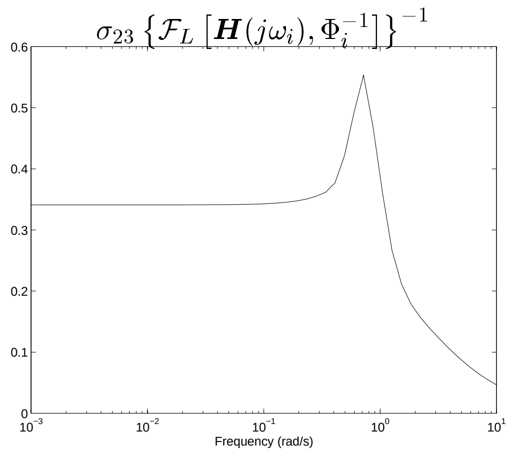

Figure 4: Test of necessary condition for consistency with $\sigma_{23}\left\{\mathcal{F}_L\left[\boldsymbol{H}(j\omega_i),\Phi_i^{-1}\right]\right\}^{-1}$.

between 0.001 rad/s and 10 rad/s were generated for the input disturbance rejection closed-loop configuration with $\boldsymbol{G}_a$ and $\boldsymbol{K}_1$. Note that Assumption (A3.1) was satisfied. The necessary condition of Theorem 2 is tested by computing $\sigma_{23}\left\{\mathcal{F}_L\left[\boldsymbol{H}(j\omega_i),\Phi_i^{-1}\right]\right\}^{-1}$ for $i = 1,\ldots,50$ and checking that all these numbers are less than 1. The results plotted in Figure 4 show that the necessary condition has been satisfied. Hence the model was not invalidated. This had to be expected since the data were generated by an admissible plant in $\mathcal{P}$.

4 Conclusion

Solutions to the closed-loop model/data consistency problem for coprime factorizations were shown to be difficult to obtain, but potentially very useful for unstable or lightly-damped systems. The noise-free closed-loop multi-input, multi-output consistency problem was studied. For a standard left-coprime factorization, a necessary condition was given as a simple test consisting of computing minimum-norm solutions to underdetermined linear complex matrix equations, as in the open-loop case. For an left-coprime factor model of an LFSS, we gave a necessary condition based on the

Schmidt-Mirsky Theorem. In both cases, the boundary interpolation theorem of [2] was invoked. Sufficient conditions for the closed-loop model/data consistency problem will be given in a separate paper.

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