3.1 Banach Spaces

A normed linear space \((V, \| \cdot \|)\) is said to be a Banach space (or complete normed linear space) if every Cauchy sequence in \((V, \| \cdot \|)\) converges to a vector of \(V\).

3.1.1 Continuity of Functions

The notion of distance in a normed linear space enables us to define the continuity of functions.

Let \((U, \| \cdot \|_U), (V, \| \cdot \|_V)\) be normed linear spaces, and consider the function \(f : U \rightarrow V\).

(a) The function \(f\) is said to be continuous at \(u_0 \in U\) if, for every \(\varepsilon > 0\), there exists a \(\delta = \delta(\varepsilon, u_0)\) such that \(\| f(u) - f(u_0) \|_V < \varepsilon\), whenever \(\| u - u_0 \|_U < \delta\).

(b) \(f\) is said to be continuous if it is continuous at all \(u \in U\).

(c) \(f\) is said to be uniformly continuous if, for every \(\varepsilon > 0\), there exists a \(\delta = \delta(\varepsilon)\) such that \(\| f(u) - f(u_0) \|_V < \varepsilon\), whenever \(\| u - u_0 \|_U < \delta\).

Proposition:

\((C[a, b], \| \cdot \|)\) is a Banach space where \(C[a, b] := \{ x : [a, b] \rightarrow \mathbb{R} \}\) is the space of continuous functions on \([a, b] \subset \mathbb{R}\), and \(\| x \| := \max_{t \in [a, b]} |x(t)|\).

Proof (sketch):

First, we need the concept of uniform convergence. A sequence of real-valued functions \(\{ f_i \}_{i=1}^{\infty}\) uniformly converges to \(f\) over \([a, b] \subset \mathbb{R}\) if, given \(\varepsilon > 0\), \(\exists N(\varepsilon)\) such that \(|f_i(t) - f(t)| < \varepsilon\), \(\forall i > N, \forall t \in [a, b]\).

Since \([a, b] \subset \mathbb{R}\) is assumed to be bounded and \(x(t)\) is continuous, the maximum of \(|x(t)|\) is well-defined and finite for every \(x \in C[a, b]\). Note that a sequence of functions \(\{ x_i \} \subset C[a, b]\) converges to a function \(x \in C[a, b]\) iff the sequence of real numbers \(\{ x_i(t) \}\) converges to \(x(t)\) uniformly \(\forall t \in [a, b]\). This uniform convergence is ensured by the use of the norm \(\| x \| := \max_{t \in [a, b]} |x(t)|\) on \(C[a, b]\).
Equipped with the $L^2$-norm $\|x\|_2 := \sqrt{\int_0^1 |x(t)|^2 \, dt}$, the space $C[a, b]$ is not complete. For example, consider the sequence $\{x_i(t)\}_{i=1}^{\infty}$ where $x_i(t) := \begin{cases} (2i)^i, & 0 \leq t \leq \frac{1}{2} \\ 1, & \frac{1}{2} < t \leq 1 \end{cases}$.

This is a Cauchy sequence: for $m > n$

$$\|x_n - x_m\|_2^2 = \int_0^1 |x_n(t) - x_m(t)|^2 \, dt \leq \int_0^{0.5} |x_n(t)|^2 \, dt \to 0 \text{ as } m \to \infty$$

However, there cannot be a continuous function $x(t)$ such that $\|x_n - x\|_2 \to 0$, e.g., $C[0, 1]$ is not big enough as it does not contain the discontinuous step function.

**Proposition: Schwarz Inequality**

Let $x, y$ be vectors of the inner product space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$. Then:

(i) $|\langle x, y \rangle| \leq \|x\| \|y\|$

(ii) $|\langle x, y \rangle| = \|x\| \|y\|$ $\iff$ the vectors $x, y$ are linearly dependent

### 3.1.2 Hilbert Spaces

**Definition:**

An inner product space that is complete in the norm induced by the inner product is called a **Hilbert space**.
Examples:

(a) Let $V = \mathbb{R}^n$, $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$. Then, $(V, \langle \cdot, \cdot \rangle)$ is a Hilbert space with norm

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

(this is the $l_2$-norm.)

(b) $V = C^n[a, b]$, where

$C^n[a, b] := \{ x : [a, b] \to \mathbb{R}^n \mid x$ is continuous, $[a, b]$ is a bounded interval of $\mathbb{R}\}$

with the inner product $\langle x, y \rangle_c := \int_{a}^{b} \langle x(t), y(t) \rangle dt$ constitutes the inner product space of continuous functions $(V, \langle \cdot, \cdot \rangle_c)$. However, with this inner product, $(V, \langle \cdot, \cdot \rangle_c)$ is not a Hilbert space. To see this, consider the discontinuous function:

$$y(t) := \begin{cases} 0, & a \leq t \leq T, \\ 1, & T < t < b. \end{cases}$$

A real Fourier series expansion of $y(t)$ is given by:

$$\tilde{y}(t) = \sum_{k=0}^{+\infty} \left[ \alpha_k \sin(n\omega_0 t) + \beta_k \cos(n\omega_0 t) \right], \quad \omega_0 = \frac{2\pi}{b-a}.$$

Then, we know that this Fourier series converges to the function $y(\cdot)$ in a mean-square sense, i.e., in the norm defined by the above inner product

$$\|y\| := \sqrt{\langle y, y \rangle_c} = \sum_{k=0}^{b-a} \int_{a}^{b} \langle y(t), y(t) \rangle dt = \int_{a}^{b} |y(t)|^2 dt.$$

Thus, the partial sums of the Fourier series constitute a Cauchy sequence in $V = C^n[a, b]$ which does not converge to an element of $V$. Hence $(V, \langle \cdot, \cdot \rangle_c)$ is not a Hilbert space because it is not complete.

3.1.2.1 $L_p$ spaces

Let $S := \{ f : \mathbb{R} \to \mathbb{R}^{m\times n} \}$ be the set of (Lebesgue) measurable functions. Note that a function is measurable if it is the limit of a sequence of staircase (piecewise constant) functions at all $t$ except those times belonging to a set of measure zero. Define the subspaces

$$S_+ := \{ f \in S : f(t) = 0, \forall t < 0 \}$$

$$S_- := \{ f \in S : f(t) = 0, \forall t < 0 \}$$

(i) Finite horizon: $L_p[0, T]$
In this case the norm is defined as \[ \|f\|_{0,T}^{0,T} := \left( \int_0^T |f(t)|^p \, dt \right)^{1/p} \] . The \( L_p[0,T] \) space is composed of functions over the interval \([0,T]\) whose norm is finite. The \( L_2[0,T] \) space is the most commonly used as it represents the class of functions of finite-energy. More precisely:

\[ L_2[0,T] := \left\{ f \in S_2 : \|f\|_{0,T}^{0,T} < \infty \right\}. \]

Note that \( L_2[0,T] \) is a Hilbert space (though not for \( p \neq 2 \)) with inner product:

\[ \langle x, y \rangle_{0,T} := \int_0^T \langle x(t), y(t) \rangle \, dt = \int_0^T x(t)^* y(t) \, dt \]

inducing the norm \( \|f\|_{0,T1,2} \).

Note that any signal that is continuous on \([0,T]\) is bounded, and therefore lies in \( L_2[0,T] \).

(ii) Infinite horizon case

We have \( L_p(-\infty, +\infty) := \left\{ f \in S : \|f\|_p < \infty \right\} \), where \( \|f\|_p := \left( \int_{-\infty}^{+\infty} |f(t)|^p \, dt \right)^{1/p} \).

Also:

\[ L_p[0,+\infty) = S_+ \cap L_p(-\infty, +\infty) \]
\[ L_p(-\infty, 0) = S_- \cap L_p(-\infty, +\infty) \]

Again, \( L_2(-\infty, +\infty) \), \( L_2[0,+\infty) \), \( L_2(-\infty, 0) \) are Hilbert spaces (not for \( p \neq 2 \)).

Feedback control problems are typically set up in \( L_2[0,+\infty) \) for the input signals, and one often seeks to prove stability by showing that all outputs are also in \( L_2[0,+\infty) \). The extended \( L_2 \) space is used for this purpose:

\[ L_2 := \left\{ f : f \in L_2[0,T], \forall 0 < T < +\infty \right\} \]

Note that \( f \in L_2 \Rightarrow \sup_T \|f\|_{0,T1,2} < \infty \), e.g., \( f(t) = t, \ g(t) = e^t \in L_2 \), but they don't belong to \( L_2[0,+\infty) \).

If \( p = \infty \), then \( L_\infty[0,+\infty) \) is the space of measurable functions that are essentially bounded on \([0,+\infty)\). The norm of \( f \in L_\infty[0,+\infty) \) is defined by:
\[ \|f\|_\infty := \text{essential supremum of } |f(t)| \]
\[ = \inf_{f = f} \sup_{t \text{ almost everywhere}} |f'(t)| \]

**Example:**

\[ f(t) := \begin{cases} 1 - t^2, & t \in [-1, 1], t \neq 0 \\ 2, & t = 0 \end{cases} \]

We have \( \sup_{t \in [-1, 1]} |f(t)| = 2 \), whereas \( \text{ess sup}_{t \in [-1, 1]} |f(t)| = \| f \|_\infty = 1 \).

(iii) **Signals in the Frequency Domain**

A frequency domain signal is a (measurable) function \( \hat{f}(j\omega) \) such that \( \hat{f}^*(j\omega) = \hat{f}^T(-j\omega) \) (conjugate transpose.)

The norm is defined by:

\[ \| \hat{f} \|_2 := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(j\omega) \hat{f}(j\omega) d\omega \right)^{1/2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \| \hat{f}(j\omega) \|^2 d\omega \right)^{1/2} \]

and the frequency-domain Lebesgue \( L_2 \) space consists of those signals with finite norm:

\[ L_2 := \left\{ \hat{f} : \| \hat{f} \|_2 < +\infty \right\}. \]

\( L_2 \) is a Hilbert space under the inner product:

\[ \langle \hat{f}, \hat{g} \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}^*(j\omega) \hat{f}(j\omega) d\omega. \]
For $f \in L_2(-\infty, +\infty)$, the Fourier transform of $f$ is:

$$\hat{f}(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt$$

Note that the Fourier transform is a Hilbert space isomorphism between $L_2(-\infty, +\infty)$ and $L_2$, i.e., it preserves the inner product and the 2-norm, e.g.,

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle,$$  
(Parseval’s Identity)

$$\|f\|_2 = \|\hat{f}\|_2.$$ 

The frequency-domain space $L_\infty$ is also useful. Its norm is defined as the maximum magnitude over all frequencies:

$$\|\hat{f}\|_\infty = \text{ess sup}_{\omega \in \mathbb{R}} |\hat{f}(j\omega)|.$$ 

Thus frequency-domain Lebesgue $L_\infty$ space consists of those signals with finite $\infty$-norm:

$$L_\infty := \left\{ \hat{f} : \|\hat{f}\|_\infty < +\infty \right\}.$$ 

(iv) $H_2$ space (Hardy 2-space)

This is the space of complex-valued functions of a complex variable which are analytic in the open right-half plane, with finite norm:

$$\|f\|_2 := \left( \sup_{\alpha > 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f^*(\alpha + j\omega) f(\alpha + j\omega) d\omega \right)^{1/2} = \left( \sup_{\alpha > 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|f(\alpha + j\omega)\|^2 d\omega \right)^{1/2}$$

i.e.,

$$H_2 = \left\{ f : f \text{ is analytic in } \text{Re}\{s\} > 0 \text{ and } \|f\|_2 < +\infty \right\}.$$ 

For any $f \in H_2$, the $H_2$-norm is actually evaluated on the $j\omega$-axis by using:

$$\|f\|_2 := \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|f_b(j\omega)\|_{L^2}^2 d\omega \right)^{1/2},$$

where $f_b(j\omega) = \lim_{\alpha \to 0} f(\alpha + j\omega)$ exists for almost all $\omega$. 

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Paley-Wiener Theorem:

\( \mathcal{H}_2 \) is isomorphic to \( \mathcal{L}_2^2(0, +\infty) \) under the Laplace transform, where for any \( f \in S \),

\[
\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt.
\]

For \( f \in \mathcal{L}_2^2(0, +\infty) \) the region of convergence of the Laplace transform is \( \text{Re}\{s\} > 0 \) and \( \hat{f} \in \mathcal{H}_2 \).

### 3.2 Induced Norms of Linear Operators

In this section, the concept of induced norm of a linear operator is introduced. The induced norm is an important quantity in linear systems theory as it measures the maximum “gain” of a system.

An LTI system, characterized by its impulse response or its transfer function, can have various induced norms, corresponding to the choice of input and output signal spaces.

#### 3.2.1 Norm of a Particular Response

One could define the norm of the linear system (linear operator) mapping \( \mathcal{X} \rightarrow \mathcal{Y} \) as the norm of the output signal given a fixed input signal:

\[
\|H\| := \|Hx_p\|_y,
\]

where the particular input \( x_p \) might be a unit impulse, step, etc.

#### 3.2.2 Worst-Case Response Norm

Here, we are interested in measuring the size of a system in terms of its maximum output norm over the input space or a subset of it. Thus the induced norm can be defined as follows:

\[
\|H\| := \sup_{x \in \mathcal{X}} \|Hx\|_y.
\]

#### 3.2.3 The Induced Norm as the Gain of a Linear System

Here, we are interested in measuring the size of a system in terms of its maximum amplification over the input space or a subset of it. The amplification is measured as the ratio of the norm of the input and the norm of the corresponding output. This is by far the most used definition of the norm of a system.
Thus the induced norm, or system gain is defined as:

$$\|H\|_{\text{gain}} := \sup_{x \in X} \sup_{x \neq 0} \frac{\|Hx\|_Y}{\|x\|_X} = \sup_{x \in X} \|Hx\|_Y.$$

(show the second equality as an exercise)

**Example:**

Suppose that we have a BIBO stable, causal LTI system, with impulse response \(h\) and transfer function \(H\), mapping \(L_2[0, +\infty)\) into \(L_2[0, +\infty)\). Recall that causality is equivalent to \(h(t) = 0, t < 0\) , and BIBO stability is equivalent to \(H\) having no closed RHP poles and a finite maximum magnitude on the \(j\omega\)-axis. What is its \(L_2\)-induced norm?

$$\|H\|_{\text{gain}}^2 = \sup_{\|H(j\omega)\|_{L_2} \leq 1} \|h * x\|_{L_2[0, +\infty)}^2$$

$$= \sup_{\|H\|_{L_2} \leq 1} \|H\tilde{x}\|_{L_2}^2$$

$$= \sup_{\|H\|_{L_2} \leq 1} \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( (H\tilde{x})(j\omega) \right)^* (H\tilde{x})(j\omega) d\omega \right)$$

$$= \sup_{\|H\|_{L_2} \leq 1} \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{x}(j\omega)^* H(j\omega)^* H(j\omega)\tilde{x}(j\omega) d\omega \right)$$

$$= \sup_{\|H\|_{L_2} \leq 1} \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{x}(j\omega)^* \left| H(j\omega) \right|^2 \tilde{x}(j\omega) d\omega \right)$$

$$\leq \sup_{\|H\|_{L_2} \leq 1} \left( \sup_{\omega} \left| H(j\omega) \right|^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \tilde{x}(j\omega) \right|^2 d\omega \right)$$

$$= \sup_{\omega} \left| H(j\omega) \right|^2 \sup_{\|H\|_{L_2} \leq 1} \|\tilde{x}\|_{L_2}^2 = \sup_{\omega} \left| H(j\omega) \right|^2$$

It can be shown that the inequality is tight for an input signal whose energy would be concentrated at the frequency where the maximum magnitude of the frequency response of the system occurs. Therefore, the \(L_2\)-induced norm of the system is actually the \(L_{\infty}\)-norm of its frequency response:

$$\|H\|_{\text{gain}} = \sup_{\omega} \left| H(j\omega) \right| = \|H\|_{\infty}.$$

This norm, when specialized to stable systems with finite \(L_{\infty}\)-norm, i.e., to the space \(H_{\infty}\), is called the \(H_{\infty}\)-norm of the system.