Lecture 24: Observability and Constructibility

7 Observability and Constructibility

Motivation:

State feedback laws depend on a knowledge of the current state. In some systems, $x(t)$ can be measured directly, e.g., position and velocity of an automobile.

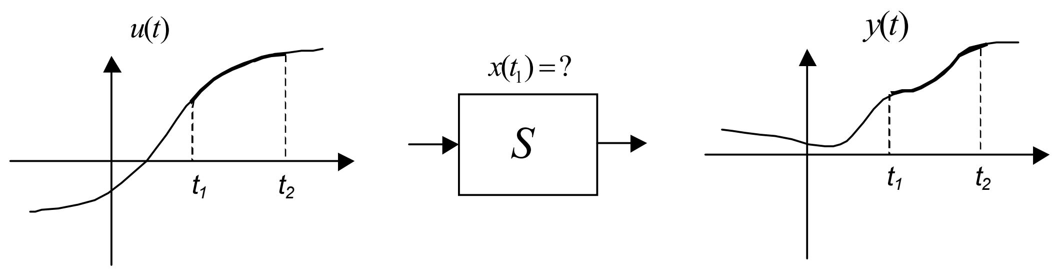
However, it is often difficult or even impossible to measure all state variables directly, e.g., the internal temperatures and pressures of a chemical process. It is therefore desirable to estimate the current state from the knowledge of the history of inputs and outputs of the system over some finite interval.

A device which accomplishes estimates of the state is called an observer, and this leads to the concepts of observability and constructibility, which are concerned with determining whether an estimate of the state can be constructed when input-output histories are known.

On the other hand, if measurements of $u(\cdot)$, $y(\cdot)$ are contaminated by noise, then the best we can do is to estimate $x(t)$ approximately, by invoking filters which achieve the best approximation as measured by some objective function, e.g., the Kalman filter.

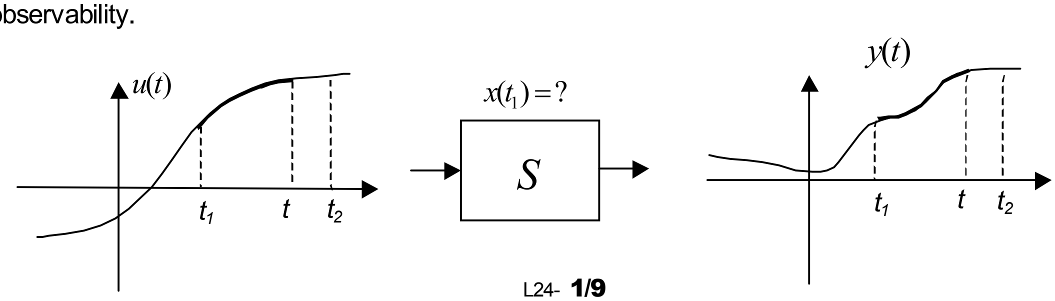
Observability:

Observability is concerned with the ability of determining the present state $x(t_1)$ from knowledge of present and *future* system output $y(t)$ and input $u(t)$, $t \in [t_1, t_2]$.



Constructibility:

Constructibility is concerned with the ability of determining the present state $x(t)$ from knowledge of present and *past* system output $y(t)$ and input $u(t)$, $t \in [t_1, t_2]$. Clearly, this is more interesting than observability.



It will be shown that observability always imply constructibilty, whereas constructibility implies observability only when the state transition matrix of the system is nonsingular.

7.1 Discrete Time Invariant Case

Consider the system governed by the differential system

$$\begin{aligned} x(j+1) &= Ax(j) + Bu(j), \quad x(r) = x_r \\ y(j) &= Cx(j) + Du(j) \end{aligned} \tag{7.1}$$

where $x(r) = x_r$ is the initial state. We are interested in uniquely determining the initial state $x(r) = x_r$ from present and future system input and output (observability problem.) The output of the system is

$$y(j) = CA^{j-r}x_r + \sum_{i=r}^{j-1} CA^{j-(i+1)}Bu(i) + Du(j), \quad j > r \,. \tag{7.2}$$

Rewrite (7.2) as follows:

$$\begin{aligned} \tilde{y}(j) &:= y(j) - \left[\sum_{i=r}^{j-1} CA^{j-(i+1)}Bu(i) + Du(j) \right], \quad j > r \\ \tilde{y}(r) &:= y(r) - Du(r) \end{aligned} \,. \tag{7.3}$$

Thus,

$$\tilde{y}(j) = CA^{j-r}x_r, \quad j > r \,. \tag{7.4}$$

and this equation can be solved by first computing $\tilde{y}(j)$, and inverting

$$\underbrace{\begin{bmatrix} \tilde{y}(r) \\ \tilde{y}(r+1) \\ \vdots \\ \tilde{y}(r+k-1) \end{bmatrix}}_{\tilde{Y}_{k-1}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{\mathcal{O}_k} x_r \tag{7.5}$$

or,

$$\tilde{Y}_{k-1} = \mathcal{O}_k x_r, \qquad \tilde{Y}_{k-1} \in \mathbb{R}^{(k-r+1)p}, \ \mathcal{O}_k \in \mathbb{R}^{(k-r+1)p \times n} \tag{7.6}$$

Clearly, the system is observable if the solution x_r of (7.5) is unique, i.e., if it is the only initial state that, together with the given input-output histories can generate the observed output history. Now, (7.6) has a unique solution $x_r \Leftrightarrow \mathcal{N}\{\mathcal{O}_k\} = \{\theta\} \Leftrightarrow \operatorname{rank}\{\mathcal{O}_k\} = n$.

Definition: Observability Matrix

The linear time-invariant system (7.1) is said to be observable if the observability operator in (7.4) which defines a matrix $\mathcal{O}_n$ has full column rank, where

$$\mathcal{O}_n := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad (7.7)$$

e.g., $\operatorname{rank}\{\mathcal{O}_n\} = n$.

Remark:

Note that by the CHT, $\mathcal{N}\{\mathcal{O}_n\} = \mathcal{N}\{\mathcal{O}_k\},\ k \geq n$, but $\mathcal{N}\{\mathcal{O}_n\} \subseteq \mathcal{N}\{\mathcal{O}_k\},\ k < n$. Therefore, in general one has to observe the output in n steps. Hence, (7.6) reduces to

$$\tilde{Y}_{n-1} = \mathcal{O}_n x_r, \qquad \tilde{Y}_{n-1} \in \mathbb{R}^{np},\ \mathcal{O}_n \in \mathbb{R}^{np\times n} \qquad (7.8)$$

For the system to be observable, the controllability matrix must have full column rank, i.e., it must be one-to-one.

A state x_1 is called *unobservable* if $\mathcal{O}_n x_1 = \theta$, and $\mathcal{N}\{\mathcal{O}_n\}$ is called the *unobservable space.*

Examples:

$$\begin{aligned} x(j+1) &= Ax(j),\ \ x(0) = x_0 \\ y(j) &= Cx(j) \end{aligned}$$

(a) $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Observability matrix $\mathcal{O}_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\operatorname{rank}\{\mathcal{O}_2\} = 2$, therefore x_1 can be uniquely determined from 2 output measurements (here the system has no input.)

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} y(1) - y(0) \\ y(1) \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Observability matrix $\mathcal{O}_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ \operatorname{rank}\{\mathcal{O}_2\} = 1$, therefore the system is not observable.

Since $\mathcal{N}\{\mathcal{O}_2\} = \left\{ \begin{bmatrix} 0 \\ \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$, such states are unobservable.

7.2 Continuous-Time Case

Consider the LTV system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_1) = x_1 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{7.9}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and A, B, C, D have piecewise continuous entries.

7.2.1 Special Case Observability

Consider first the special case of estimating the state $x(t_1)$ from observing the output on the interval $[t_1, t_2]$ when $u(t) = 0$.

The restriction of the zero-input response $y(\cdot)$ to $[t_1, t_2]$, denoted as $y_1(\cdot)$ is called the *tail* of the output. The observer equation relates the state to the tail:

$$y_1(t) = C(t)\Phi(t, t_1)x_1, \ \ \forall t \in [t_1, t_2] \tag{7.10}$$

Thus, the observability operator, which maps states at $t = t_1$ into tails, is the operator defined by:

$$\begin{aligned} &£ : \mathbb{R}^n \to \mathcal{L}_2[t_1, t_2] \\ &(£x_1)(t) = C(t)\Phi(t, t_1)x_1 \end{aligned} \tag{7.11}$$

The operator form of (7.10) is:

$$£x_1 = y_1 \tag{7.12}$$

<u>Definition: Complete Observability</u>

The system $(A(\cdot), C(\cdot))$ is *completely observable* (C.O.) on $[t_1, t_2]$ if the observability operator $£$ is one-to-one, i.e., if x_1 is completely determined by the tail y_1.

7.2.2 General Observability

Definition: Complete Observability

The system $\left(A(\cdot), B(\cdot), C(\cdot), D(\cdot)\right)$ is *completely observable* (C.O.) on $[t_1, t_2]$ if, given the history of $u(\cdot),\ y(\cdot)$ on $[t_1, t_2]$, $x(\cdot)$ is completely determined.

The output equation is:

$$\begin{aligned} y(t) &= C(t)\Phi(t,t_1)x_1 + \int_{t_1}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau \\ &= \underbrace{C(t)\Phi(t,t_1)x_1}_{y_1(t)\ \text{(tail)}} + \underbrace{C(t)\left(\mathcal{L}u\right)(t) + D(t)u(t)}_{y_u(t)\ \text{(forced)}} \end{aligned} \tag{7.13}$$

The general observer equation is

$$y_1 := y - y_u = \mathcal{L}x_1 \tag{7.14}$$

which relates states and tails, as in the special case.

Notes:

(a) The effect of past $(t < t_1)$ input on future $(t > t_1)$ output can be summarized by either $x(t_1)$ or the tail after t_1.

(b) The state acts as a "tag" on the tail by uniquely determining it. The converse is not generally true as a tail may have been produced by many states $x(t_1)$ for a system that is not completely observable.

(c) The problem of C.O., e.g., given $u(\cdot),\ y(\cdot)$ on $[t_1, t_2]$, find $x(\cdot)$ can be reduced to the problem of finding the initial state $x(t_1)$ from present and future input and output because $x(s) = \Phi(s, t_1)x_1,\ \ s \in [t_1, t]$. Moreover, the constructibility problem of estimating the current state $x(t)$, $t \in [t_1, t_2]$ from present and past input and output yields the solution of the C.O. problem via $x(s) = \Phi^{-1}(t, s)x(t),\ \ s \in [t_1, t]$. Clearly, constructibility and C.O. are equivalent for continuous-time LTV systems, but not necessarily for discrete-time systems because the state transition matrix is not always invertible.

(d) The C.O. condition that $\mathcal{L}$ is one-to-one is equivalent to the statement that $\mathcal{N}\{\mathcal{L}\} = \{\theta\}$. If $\mathcal{N}\{\mathcal{L}\} \neq \{\theta\}$, then the dimension of the state space is higher than is needed to determine future behavior, and there is redundancy.

The condition $\mathcal{N}\{\mathcal{L}\} = \{\theta\}$ suggests the following equivalent definition of complete observability.

Definition: Unobservable State

In a linear system, the state x_1 is *unobservable at time* t_1 if the zero-input response (tail) of the system is zero for $t \geq t_1$. That is, a state is unobservable if it is nonzero and it produces a zero tail.

Definition: Unobservable Subspace

The *unobservable subspace at time* t_1, $\mathcal{R}_{\bar{O}}^{t_1}$, is

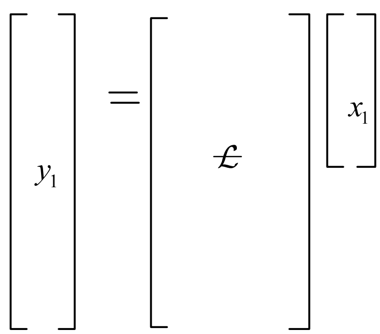
$$\mathcal{R}_{\bar{O}}^{t_1} := \left\{ x_1 \in \mathbb{R}^n : x_1 \text{ is unobservable at } t_1 \right\}. \tag{7.15}$$

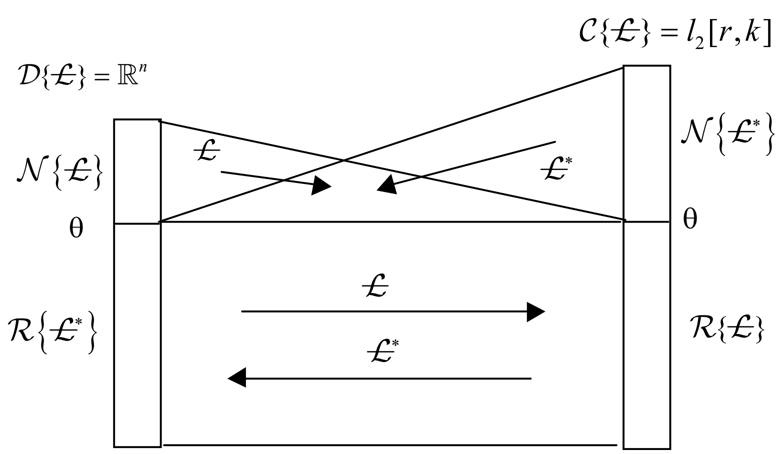
Definition: Complete Observability (alternative)

The system (7.9) is completely observable at t_1 if $\mathcal{R}_{\bar{O}}^{t_1} = \{\theta\}$.

7.3 Observability and the Structure of the Observability Operator $\mathcal{L}$

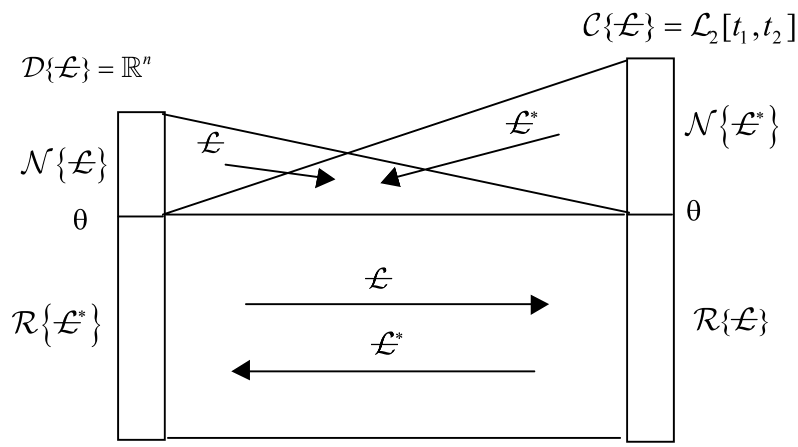
The discrete-time case is straightforward as Equation (7.8) is a finite-dimensional vector equation in which $\mathcal{L}$ is a tall matrix, as depicted below.





The observability equation does not have a solution if $y_1 \notin \mathcal{R}\{\pounds\}$, i.e., when the measured output is noisy. In this case, we can set up the problem of minimizing $\|y_1 - \pounds x_1\|$ over the possible initial state vectors (more on this later.)

The continuous-time case has the following structure diagram:



The above remark holds in the continuous-time case as well: The observability equation does not have a solution if $y_1 \notin \mathcal{R}\{\pounds\}$, but $\|y_1 - \pounds x_1\|$ can be minimized.

7.4 The Optimal Observer Problem in $\mathcal{L}_2$

Suppose the observed tail $y_1 + v$ is contaminated by a noise v. Then, the observed tail may be outside the $\mathcal{R}\{\pounds\}$, in which case the observability equation $\pounds x_1 = y_1 + v$ has no solution.

Definition: Observer Problem

Suppose the system $[A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$ is C.O. on $[t_1, t_2]$. Find x_1 to minimize the norm

$$\left\| \pounds x_1 - (y_1 + v) \right\|_{\mathcal{L}_2[t_1, t_2]}, \tag{7.16}$$

i.e., we are interested in computing x_1 which produces the tail $\pounds x_1$ that best approximates $y_1 + v$.

Note that, it is the input-output behavior rather than the state that finally matters to us, and therefore we measure the quality of approximation by how far the tail $\pounds x_1$ is from the observed $y_1 + v$ (rather than try to minimize the distance between the true and estimated states in the state-space $\mathbb{R}^n$.)

Theorem:

Suppose the system $\left[A(\cdot),B(\cdot),C(\cdot),D(\cdot)\right]$ is C.O. on $\left[t_1,t_2\right]$. Then the minimization problem

$$\min_{x_1\in\mathbb{R}^n}\left\|\mathcal{L}x_1-\left(y_1+v\right)\right\|_{\mathcal{L}_2\left[t_1,t_2\right]}, \tag{7.17}$$

on the interval $\left[t_1,t_2\right]$ has the unique solution $\left(x_1\right)_{opt}=\left(\mathcal{L}^*\mathcal{L}\right)^{-1}\mathcal{L}^*\left(y_1+v\right)$

Proof:

By the structure diagram, $\mathcal{L}_2\left[t_1,t_2\right]$ can be decomposed into a sum of two orthogonal subspaces:

$$\mathcal{L}_2\left[t_1,t_2\right]=\mathcal{R}\left(\mathcal{L}\right)\overset{\perp}{\oplus}\mathcal{N}\left(\mathcal{L}^*\right).$$

Thus, y_1+v can be decomposed into components lying in these same subspaces

$$y_1+v=\left(y_1+v_R\right)+v_N,v_R\in\mathcal{R}\left(\mathcal{L}\right),v_N\in\mathcal{N}\left(\mathcal{L}^*\right) \tag{7.18}$$

By the Projection Theorem y_1+v_R is the unique vector in $\mathcal{R}\left(\mathcal{L}\right)$ closest to y_1+v.

Since $y_1+v_R\in\mathcal{R}\left(\mathcal{L}\right)$, there exists $\left(x_1\right)_{opt}\in\mathbb{R}^n$ such that $\mathcal{L}\left(x_1\right)_{opt}=y_1+v_R$. There cannot be two solutions since $\mathcal{L}$ is assumed to be one-to-one, and hence $\left(x_1\right)_{opt}$ is unique.

Then,

$$\begin{aligned}\mathcal{L}^*\mathcal{L}\left(x_1\right)_{opt}&=\mathcal{L}^*\left(y_1+v_R\right)\\&=\mathcal{L}^*\left(y_1+v_R\right)+\mathcal{L}^*v_N\ ,\\&=\mathcal{L}^*\left(y_1+v\right)\end{aligned} \tag{7.19}$$

and since $\mathcal{L}^*\mathcal{L}$ is invertible (assuming that the system is C.O.), we have:

$$\left(x_1\right)_{opt}=\left(\mathcal{L}^*\mathcal{L}\right)^{-1}\mathcal{L}^*\left(y_1+v\right). \tag{7.20}$$

■

Remarks:

(a) The expression $\left(\mathcal{L}^*\mathcal{L}\right)^{-1}\mathcal{L}^*$ defines the pseudoinverse of any one-to-one operator $\mathcal{L}$.

(b) The noise component $v_N \in \mathcal{N}\left(\mathcal{L}^*\right)$ is filtered out by the observer, but the component $v_R \in \mathcal{R}\left(\mathcal{L}\right)$ cannot be distinguished from the true tail and remains as an error in the estimate of the true tail.

7.4.1 Evaluation of the Pseudoinverse

Recall that:

$$\begin{aligned} &\mathcal{L}:\mathbb{R}^n \to \mathcal{L}_2[t_1,t_2] \\ &\left(\mathcal{L}x_1\right)(t) = C(t)\Phi(t,t_1)x_1 \end{aligned} . \tag{7.21}$$

The adjoint $\mathcal{L}^* : \mathcal{L}_2[t_1,t_2] \to \mathbb{R}^n$ is obtained as:

$$\mathcal{L}^* y := \int_{t_1}^{t_2} \Phi^*(\tau,t_1)C^*(\tau)y(\tau)d\tau \tag{7.22}$$

<u>Definition: Backward Observability Grammian</u>

The *backward observability Grammian* is defined as:

$$N(t_1,t_2) := \mathcal{L}^*\mathcal{L} = \int_{t_1}^{t_2} \Phi^*(\tau,t_1)C^*(\tau)C(\tau)\Phi(\tau,t_1)d\tau \tag{7.23}$$

The optimal observer (or reconstruction) law is given by:

$$\left(x_1\right)_{opt} = N^{-1}(t_1,t_2)\int_{t_1}^{t_2} \Phi^*(\tau,t_1)C^*(\tau)\left(y_1(\tau)+v(\tau)\right)d\tau \; . \tag{7.24}$$

where

$$y_1(t) := \underbrace{y(t)}_{\text{measured output}} - \underbrace{y_u(t)}_{\text{zero-state response}} \tag{7.25}$$