Steerable Kernels for Arbitrarily-Sampled Spaces

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Introduction

- Steerable kernels are efficient, $O(k)$, $k << N$.
  - Perform expensive inner products once.
  - Inexpensive weighted sum of inner products span family of desired kernels.
- Many basis functions determined *ad hoc*.
  - (e.g. Specific for oriented edges)
  - Correspondence matrix formalizes discrete mapping.
  - Optimal real-valued basis functions.
    - Tesselation of data space (square, hexagonal, ...)
    - Mapping class (rotation, scaling, ...)

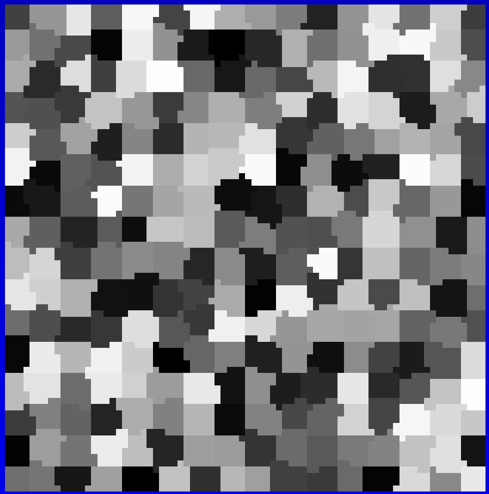
Steerable functions

$$F_\alpha(\mathbf{x}) \;=\; \sum_{k=1}^{N} b_k(\alpha) G_k(\mathbf{x})$$

- Steerable function $F_\alpha(x)$
- Data vector x
- Deformation parameter α
- Basis functions $G_k(x)$
- Interpolation functions $b_k(\alpha)$

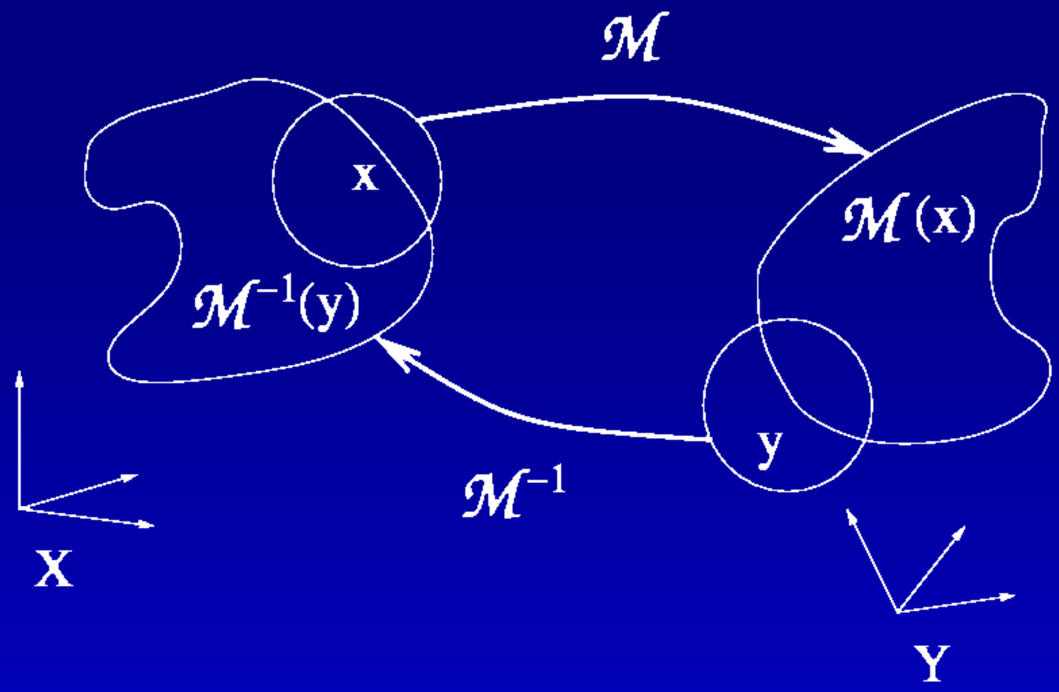
Steerable functions for rotation

- Rotation about center of aperture.
  - Only pixels within inscribed circle can fully rotate.
- Square pixels in windows of 32x32 pixels.
- Hexagonal pixels in windows of ~ 32x32 pixels.
- Irregularly-spaced Voronoi cells in 16x16 grid.
  - Voronoi cells' positions perturbed randomly by 0% to 20% of diameter.
  - Can useful basis images still be computed?



Correspondence matrix $\mathbf{H}$

Encode mapping between each pixel in first view $<I_0>_i$ and each pixel in second view $<I_\alpha>_j$.



$$\langle \mathbf{H} \rangle_{ij} \triangleq \mathrm{P}\left([\mathbf{X}]_i \overset{\mathcal{M}}{\Longleftrightarrow} [\mathbf{Y}]_j \right) \subset [0,1]$$

$$= \frac{\left(\begin{array}{c} \mathcal{A}\left(\mathcal{M}([\mathbf{X}]_i) \cap [\mathbf{Y}]_j \right) \\ + \mathcal{A}\left([\mathbf{X}]_i \cap \mathcal{M}^{-1}([\mathbf{Y}]_j) \right) \end{array} \right)}{\mathcal{A}([\mathbf{X}]_i) + \mathcal{A}\left([\mathbf{Y}]_j \right)}$$

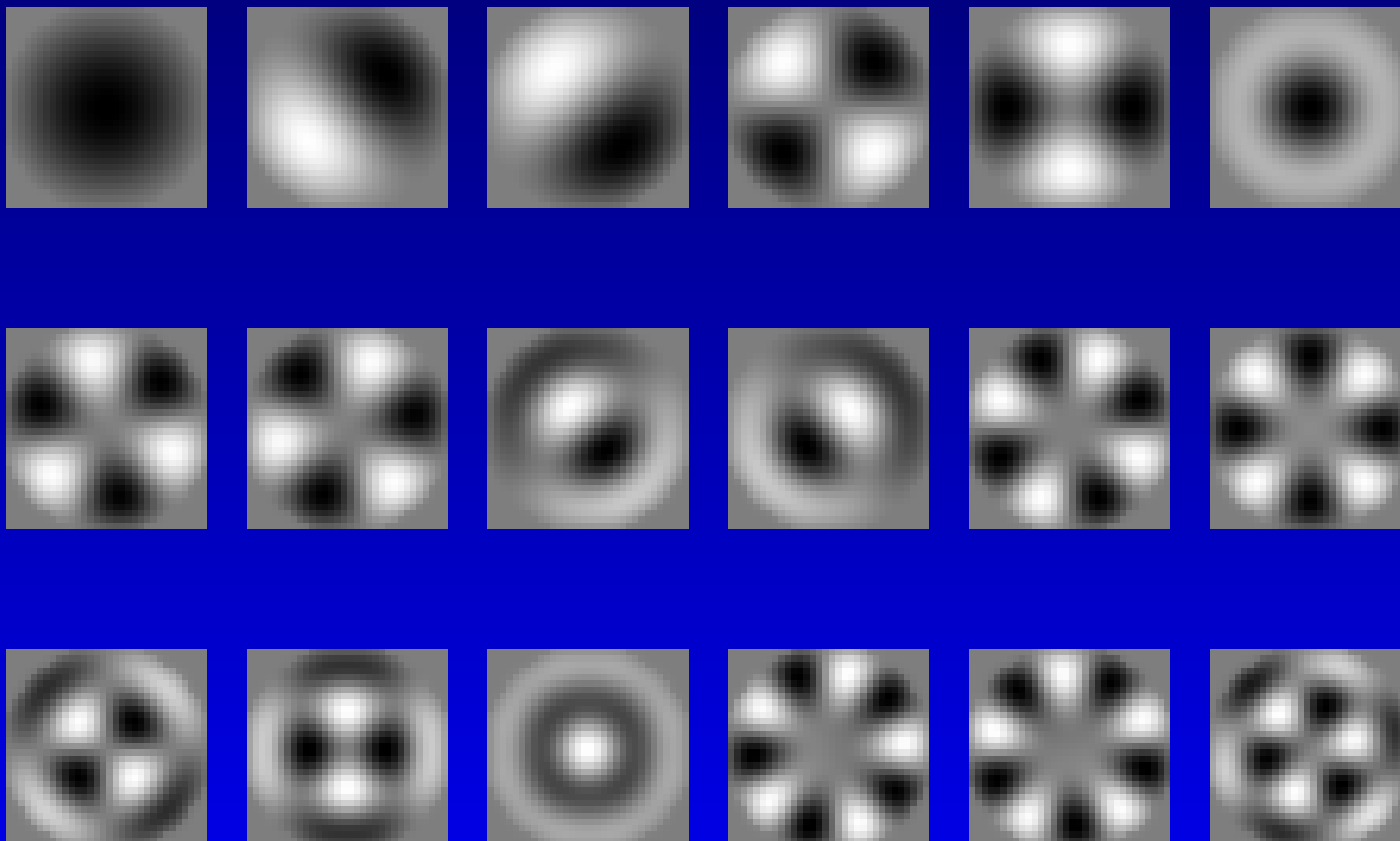
Decomposing the correspondence matrix

$$\mathbf{H}_\alpha \overset{\text{SVD}}{=} \mathbf{U}_0 \mathbf{\Sigma}_0 \mathbf{V}_\alpha{}^T$$

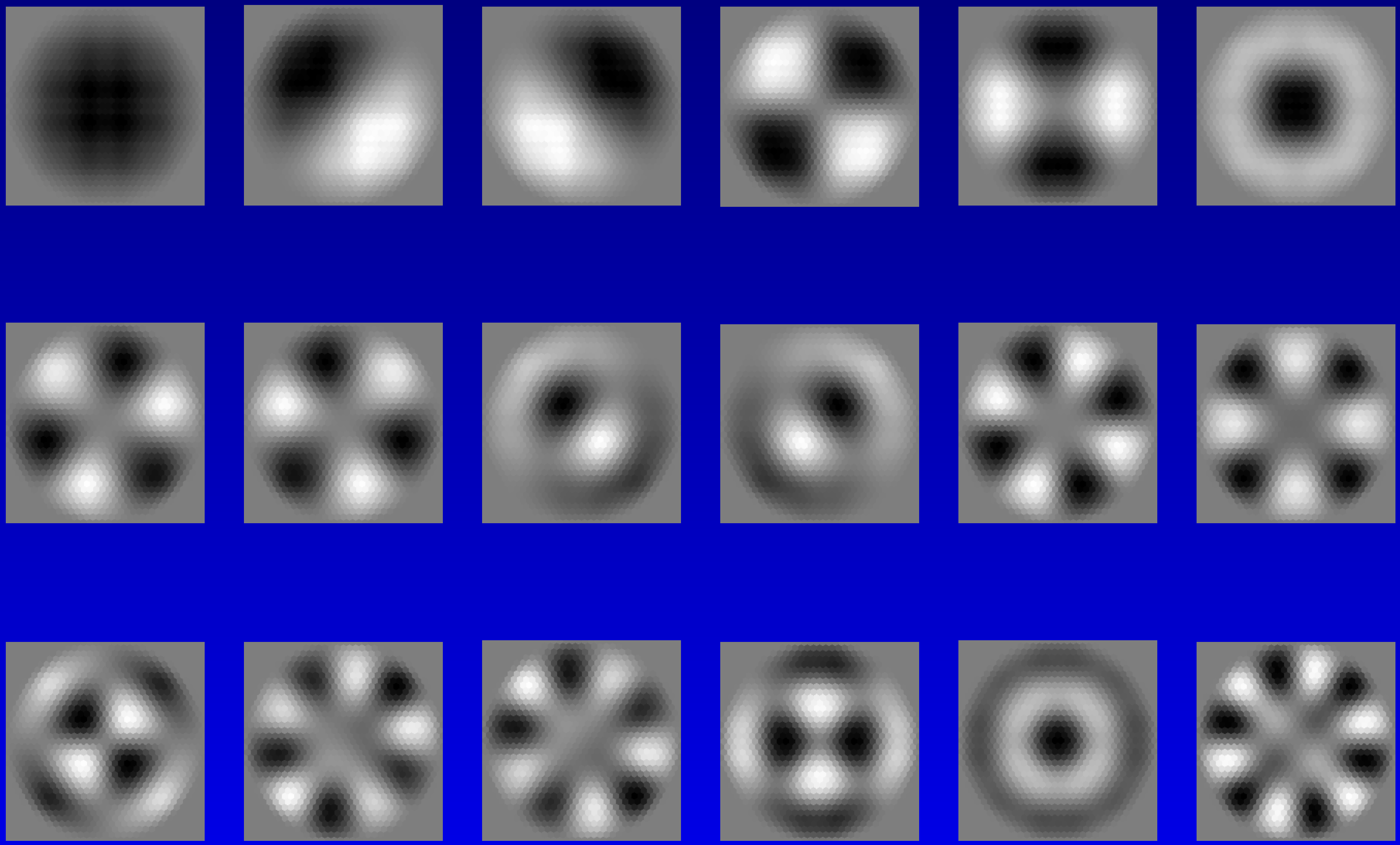
$$\mathcal{I}_0 = \mathbf{\Phi}_\alpha \mathcal{I}_\alpha$$
$$\mathcal{I}_\alpha = \mathbf{\Phi}_\alpha{}^T \mathcal{I}_0$$

$$\mathcal{I}_0 = \mathbf{U}_0 \mathbf{V}_\alpha{}^T \mathcal{I}_\alpha$$
$$\mathbf{U}_0{}^T \mathcal{I}_0 = \underset{r \times 1}{\vec{w}} = \mathbf{V}_\alpha{}^T \mathcal{I}_\alpha$$

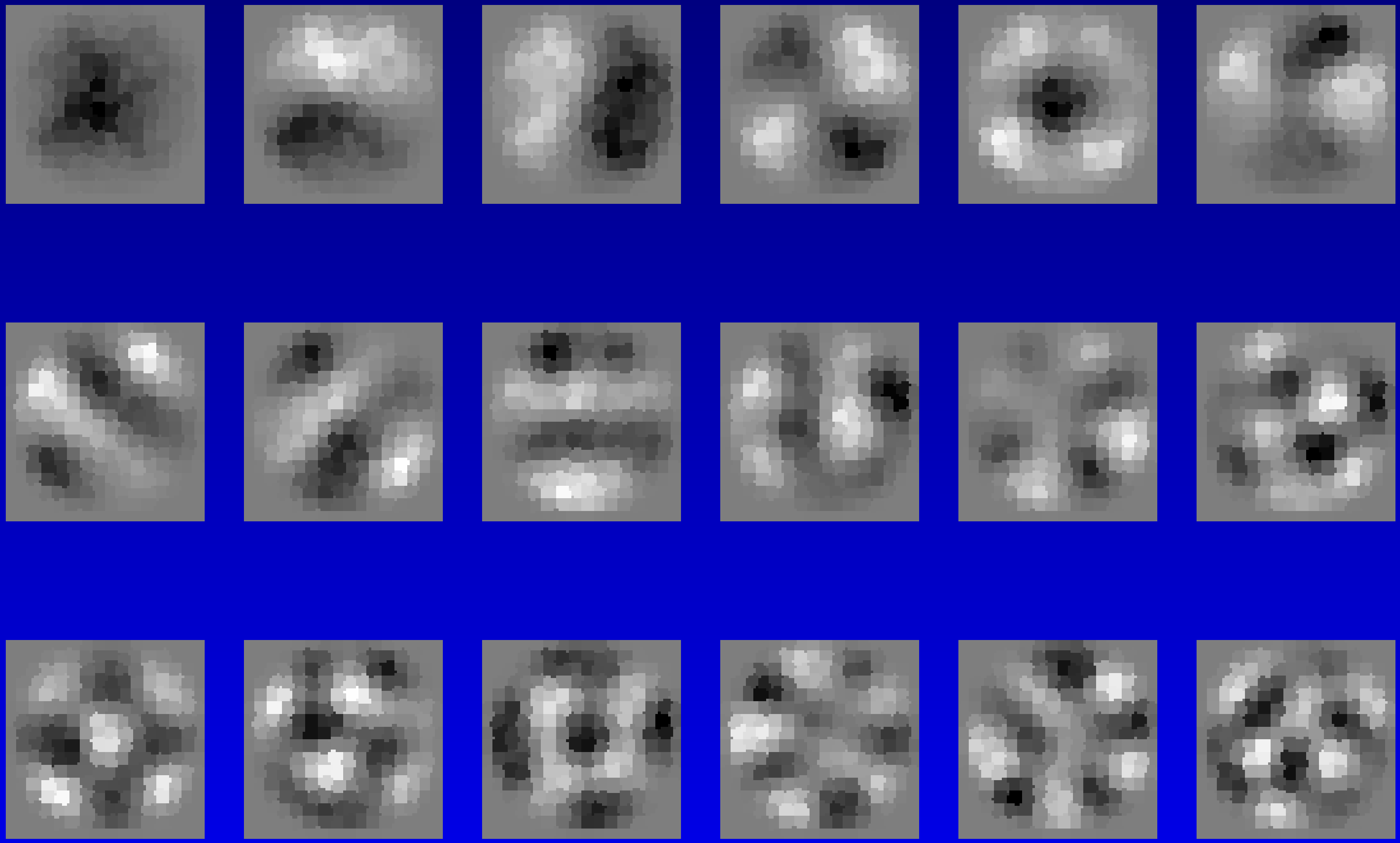
Rectilinear rotation basis fields



Hexagon rotation basis fields



Perturbed rotation basis fields



k-th order approximation

$$\mathcal{I}_0 \approx \hat{\mathbf{U}}_0 \hat{\mathbf{V}}_\alpha^T \mathcal{I}_\alpha$$

$$\hat{\mathbf{U}}_0^T \mathcal{I}_0 = \underset{k \times 1}{\hat{w}} = \hat{\mathbf{V}}_\alpha^T \mathcal{I}_\alpha$$

Solving V_α without SVD

$$\mathbf{H}_\alpha \overset{\text{SVD}}{=} \mathbf{U}_0 \mathbf{\Sigma}_0 \mathbf{V}_\alpha{}^T$$

$$\begin{aligned} \mathbf{V}_\alpha &= \mathbf{H}_\alpha{}^T \mathbf{U}_0 \mathbf{\Sigma}_0{}^+ , \\ \langle \Sigma_0{}^+ \rangle_{ij} &= \begin{cases} 1/\langle \Sigma_0{}^+ \rangle_{ij} & , i = j, i, j <= r \\ 0 & \text{, otherwise .} \end{cases} \end{aligned}$$

V resembles U : deformation interpolants

$$\underset{N \times k}{\hat{\mathbf{V}}_\alpha} = \underset{N \times k}{\hat{\mathbf{U}}_0} \, \underset{k \times k}{\hat{\mathbf{R}}_\alpha}$$

$$\hat{\mathbf{R}}_\alpha = \hat{\mathbf{U}}_0^T \hat{\mathbf{V}}_\alpha = \hat{\mathbf{U}}_0^T \mathbf{H}_\alpha{}^T \hat{\mathbf{U}}_0 \hat{\mathbf{\Sigma}}_0^+$$

$$\hat{w}_0 = \hat{\mathbf{U}}_{\mathbf{0}}^T \mathcal{I}_0 \quad , \quad \hat{\mathcal{I}}_0 = \hat{\mathbf{U}}_0 \hat{w}_0$$

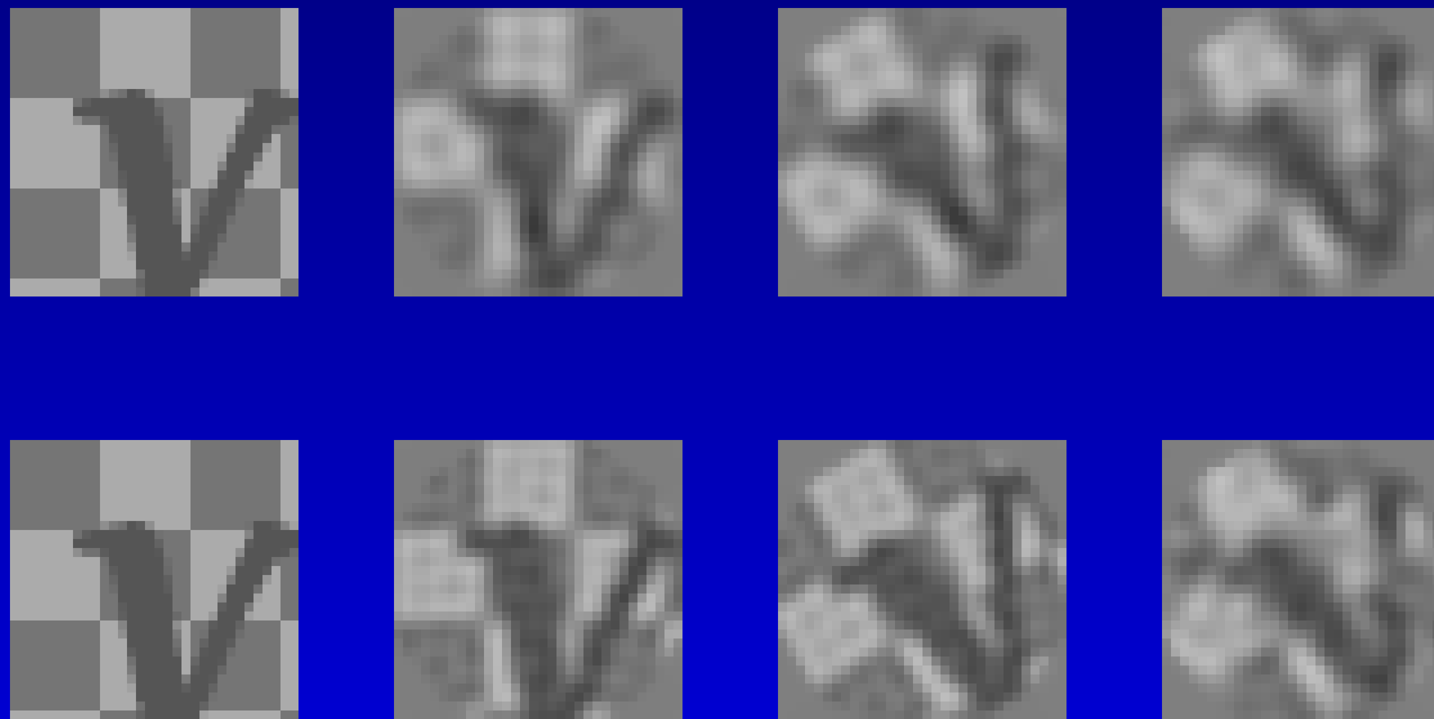
$$\hat{\mathcal{I}}_\alpha = \hat{\mathbf{V}}_\alpha \hat{w}_0 \quad , \quad \tilde{\mathcal{I}}_\alpha = \tilde{\mathbf{V}}_\alpha \hat{w}_0 = \hat{\mathbf{U}}_0 \tilde{\mathbf{R}}_\alpha \hat{w}_0$$

Rectilinear rotation experiment

Rotation α = 30 degrees. 1024 pixels.

k=100

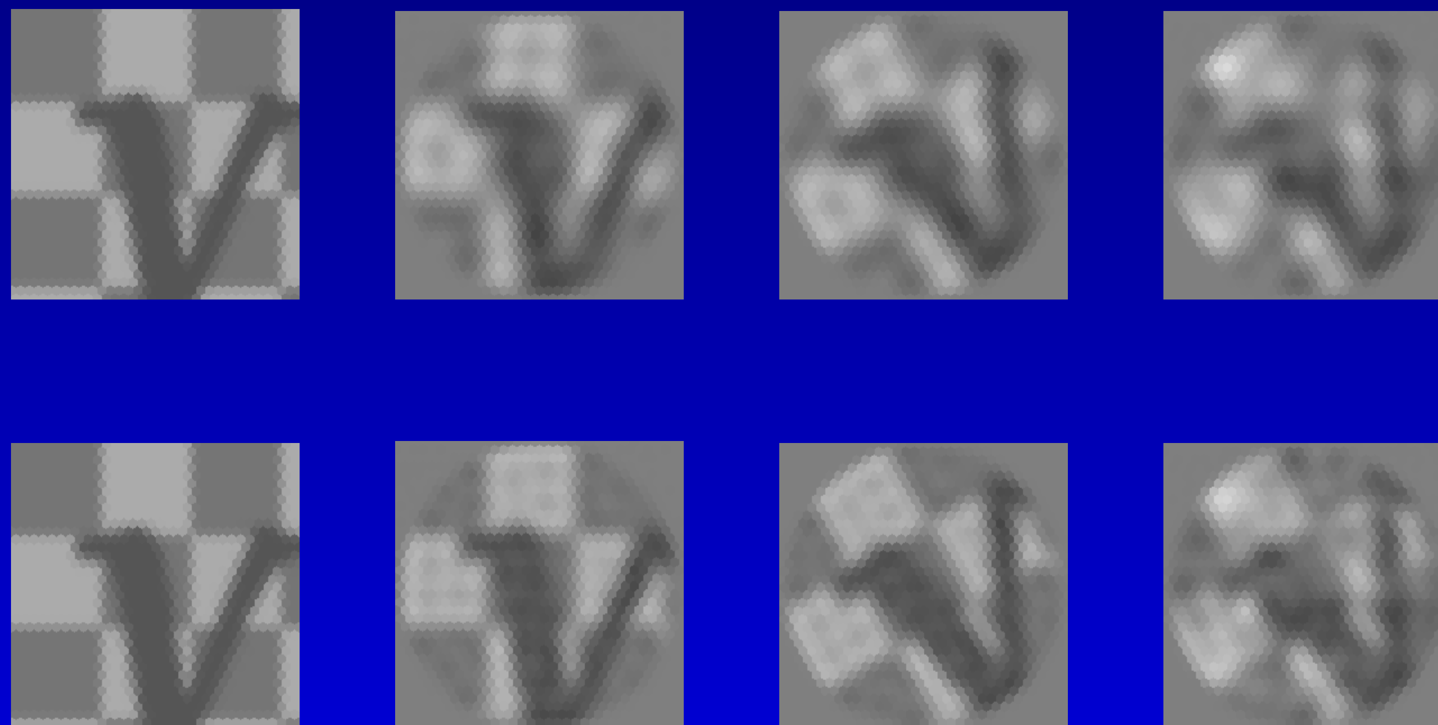
k=200



$\mathcal{I}_0$ $\hat{\mathcal{I}}_0$ $\hat{\mathcal{I}}_\alpha$ $\tilde{\mathcal{I}}_\alpha$

Hexagon rotation experiment

Rotation α = 30 degrees. 1202 pixels.



k=100

k=200

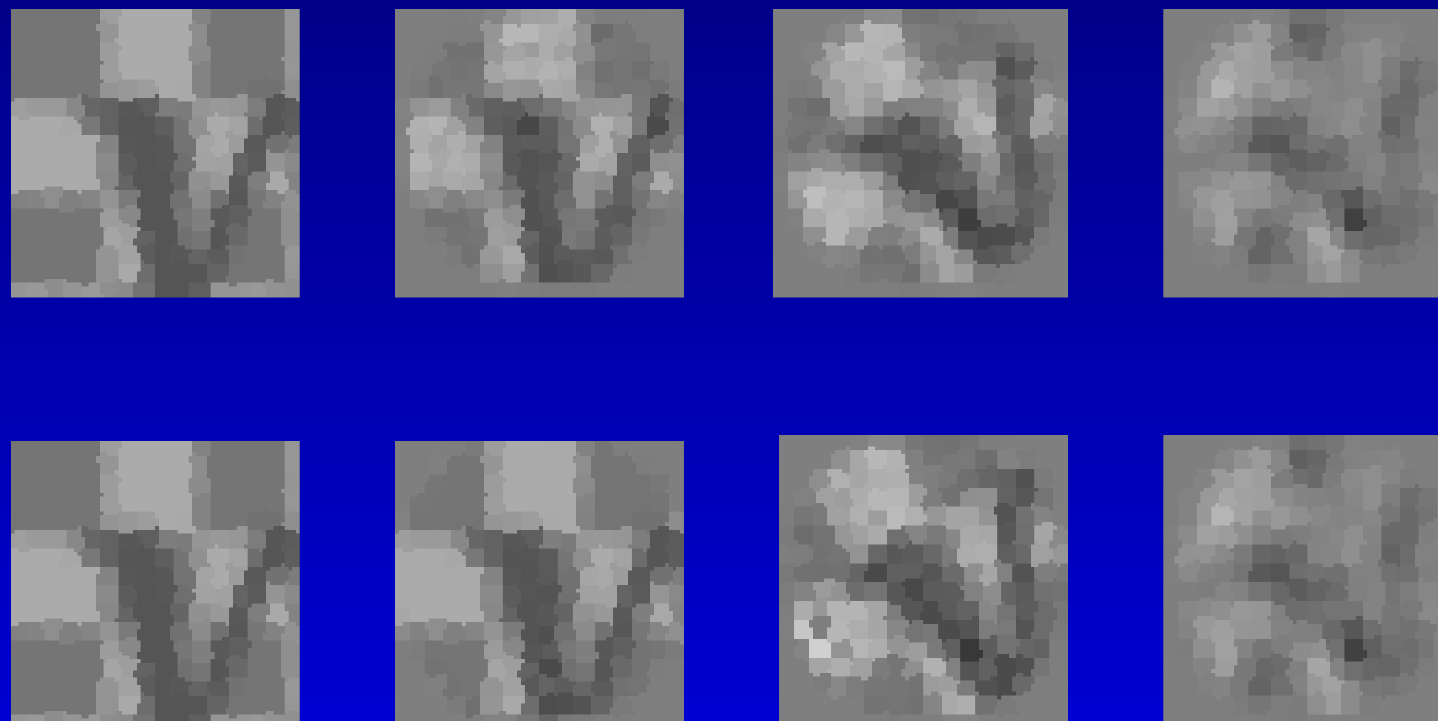
$\mathcal{I}_0$ $\hat{\mathcal{I}}_0$ $\hat{\mathcal{I}}_\alpha$ $\tilde{\mathcal{I}}_\alpha$

Perturbed rotation experiment

Rotation α = 30 degrees. 256 pixels.

k=100

k=200



$\mathcal{I}_0$ $\hat{\mathcal{I}}_0$ $\hat{\mathcal{I}}_\alpha$ $\tilde{\mathcal{I}}_\alpha$

Future work

- Formalize Degrees of Freedom of mapping ***M***.
  - Limits information conserved by tesselation, transform.
  - DOF($\boldsymbol{M}$) = rank($\mathbf{H}$).
- 3-D steerable kernels.

Conclusion

- Treat pixels as Voronoi cells.
- Image deformations between sets of Voronoi cells.
  - Without projection into rectilinear space.
- Predicts optimal real valued spectral decomposition.
- Arbitrary sampling of image space.

Thanks