Team Optimal Decentralized State Estimation of Linear Stochastic Processes by Agents with Non-Classical Information Structures *

Mohammad Afshari, Aditya Mahajan

Department of Electrical and Computer Engineering, McGill University

Abstract

We consider the problem of team optimal decentralized estimation of a linear stochastic process by multiple agents. Each agent receives a noisy observation of the state of the process and delayed observations of its neighbors (according to a pre-specified, strongly connected, communication graph). Based on their observations, all agents generate a sequence of estimates of the state of the process. The objective is to minimize the total expected weighted mean square error between the state and the agents' estimates over a finite horizon. In centralized estimation with weighted mean square error criteria, the optimal estimator does not depend on the weight matrix in the cost function. We show that this is not the case when the information is decentralized. The team optimal decentralized estimates depend on the weight matrix in the cost function. In particular, we show that the optimal estimate consists of two parts: a common estimate which is the conditional mean of the state given the common information and a correction term which is a linear function of the offset of the local information from the conditional expectation of the local information given the common information. The corresponding gain depends on the weight matrix as well as on the covariance between the offset of agents' local information from the conditional mean of the local information given the common information. We show that the local and common estimates can be computed from a single Kalman filter and derive recursive expressions for computing the offset covariances and the estimation gains.

Keywords: Decentralized Linear Quadratic Gaussian Systems, Static teams, Decentralized State Estimation, Kalman Filtering.

1. Introduction

The separation of estimation and control is one of the most celebrated results in centralized stochastic control of linear systems with quadratic per-step cost and Gaussian disturbances (which is called the LQG (linear, quadratic, and Gaussian) setup). In particular, the optimal control action is equal to a gain matrix multiplied by the current state estimate. The computation of the gain and the state estimate are separated from each other. The gain matrix is computed by solving a backward Riccati equation which depends on the per-step cost and the covariance of the plant disturbance. The state estimate is updated using a Kalman filter, where the filtering gain is computed by solving a forward Riccati equation which depends on the covariances of the plant disturbance and the observation noise. The key feature of the result is that the backward Riccati equation for computing the controller gain and the forward Riccati equation for computing the filtering gain do not depend on each other and can be solved separated. This feature is sometimes called the two-way separation between estimation and control.

These simplifications do not hold for decentralized control of LQG systems. In general, non-linear control strategies may outperform the best linear strategy [1]. Even if we arbitrary restrict attention to linear control strategies, the best linear strategy may not have a finite dimensional representation [2]. Linear strategies are known to be optimal only for specific information structures [3, 4] but even in these cases, there is no general method to identify a sufficient statistics for the optimal controller. Thus, a priori, there is no separation of estimation and control.

There are a few specific models of decentralized LQG systems with partially nested information structure where the optimal controller is linear function of a sufficient statistics [5–12]. But even in these cases, the two way separation between estimation and control does not hold. The motivation of this article is to understand the reason for this lack of separation between estimation and control and explain why it is possible to identify sufficient statistics in the absence of separation.

*This research was supported by the Natural Science and Engineering Research Council of Canada (NSERC). A preliminary version of this paper was presented in the 2018 IEEE Conference on Decision and Control (CDC).

Email addresses: mohammad.afshari2@mail.mcgill.ca (Mohammad Afshari), aditya.mahajan@mcgill.ca (Aditya Mahajan)

To understand this lack of separation, we consider team optimal decentralized state estimation, which may be viewed as a dual of team optimal decentralized stochastic control. Our main result is to show that team optimal decentralized state estimation is fundamentally different from centralized state estimation and this difference provides part of the explanation for the structure of team optimal decentralized stochastic control strategies in the absence of separation.

Decentralized state estimation is an important problem in its own right and is a key component of many large scale systems including wireless sensor networks [13], power systems [14–16], vehicle platooning [17–19], and networked control systems [17]. Due to its importance in applications, several variations of decentralized estimation have been considered in the literature. Broadly speaking, the literature on decentralized estimation may be classified into three categories, which we briefly summarize below.

The first category consists of models where nodes or agents generate local estimates and then share these estimates along with some additional information to obtain centralized estimates. Decentralized estimation of interconnected subsystems with completely decentralized information structure is considered in [20]. A similar setup with noisy information exchange between the subsystems is considered in [21]. Decentralized control and estimation with one-step delay sharing information structure is considered in [22] and it is shown that the amount of transmitted data can be reduced by transmitting two vectors: a linear function of the local estimates and a linear function of the observations. A similar result for hierarchical estimation (or decentralized estimation with a fusion center) is considered in [23]. A successive orthogonalization technique for hierarchical estimation is presented in [24]. A general methodology for combining local estimates generated using different observations at different time is presented in [25]. Hierarchical estimation with delayed inter-agent communication is considered in [26, 27]. In all of the above papers, linear dynamics and Gaussian disturbances are assumed. Hierarchical estimation for general Markov processes is considered in [28].

The second category consists of models where agents are connected over (a possibly time-varying) communication graph and share their observations with their local neighbors. Each agent runs a local Kalman filter, and between each prediction step of the Kalman filter, uses one or multiple rounds of a consensus algorithm to obtain an average of the estimates of all agents. Conditions under which the local estimates agree asymptotically and the local estimation errors are stable are identified in [29–33]. A related question of asymptotic agreement in distributed estimation with consistent or inconsistent beliefs is considered in [34–37].

The third category consists of models where each agent generates an estimate of the state and the objective is to minimize a team cost which is a weighted quadratic function of the estimation errors of all agents. Such a model was first considered in [38] who considered agents with no inter-agent communication. A variation where agents have an option to choose among one of multiple observation channels was considered in [39]. In both these papers, the optimal estimation strategy was derived by lifting the state and observation to a higher dimensional space and using a standard Kalman filter in that space.

The first two categories of results do not consider team optimal state estimation. Therefore, they are not directly provide insight to separation of estimation and control in decentralizes stochastic control. The third category of results are relevant but [38, 39] consider models with no intra-agent communication which restricts their applicability.

In this paper, we consider a model belonging to the third category where agents share their observations with their neighbors over a pre-specified communication graph. Due to the information sharing, the solution approach of [38, 39] doesn't work. Instead, we obtain the optimal estimators by using results from static team theory [40] with the common information approach [41]. A detailed comparison of our results with those of [38, 39] is presented in Section 3.3.

A remarkable feature of the team optimal decentralized state estimate considered in this paper is that an agent's team optimal decentralized estimate is not the conditional mean of the state given the information at the agent; rather it is a linear function of the local information where the gain depends on the weight function of the mean square error. This makes team optimal decentralized estimation fundamentally different from centralized state estimation. This difference is explained in details in Sec 1.1. We argue in Appendix B that this feature partly explains the lack of separation between estimation and control in decentralized stochastic control and also explain the structure of optimal strategies.

1.1. Centralized vs decentralized state estimation

First consider a centralized (one-stage) state estimation problem. Let $x \in \mathbb{R}^{d_x}$, $x \sim \mathcal{N}(0, \Sigma_x)$, denote the state of a system. An agent observes $y \in \mathbb{R}^{d_y}$, where $y = Cx + v$, where C is a $d_y \times d_x$ matrix and $v \in \mathbb{R}^{d_y}$, $v \sim \mathcal{N}(0, R)$, is independent of x. The objective is to choose an estimate $\hat{z} \in \mathbb{R}^{d_z}$ of the state according to $\hat{z} = g(y)$ (where g can be any measurable function) to minimize

$$\mathbb{E}[(Lx - \hat{z})^\intercal S(Lx - \hat{z})],$$

where S is a $d_z \times d_z$ dimensional positive definite matrix and L is a $d_z \times d_x$ matrix. It is well known that the optimal estimate is given by L times the conditional mean $\hat{x}$ of the state given the observation, i.e.,

$$\hat{z} = L\hat{x}, \quad \text{where } \hat{x} := \mathbb{E}[x|y].$$

Alternatively, the optimal estimate may be written as a linear function of the observation y, i.e.,

$$\hat{z} = LKy, \quad \text{where } K = \Sigma_x C^\intercal (C\Sigma_x C^\intercal + R)^{-1}$$

It is worth highlighting the fact that *the optimal estimate does not depend on the weight matrix* S. It is perhaps for this reason that most standard texts on state estimation assume that the weight matrix $S = I$. However, when it comes to decentralized state estimation, the weight matrix S plays an important role.

To see this, consider a two-agent (one-stage) team optimal decentralized state estimation problem. Let $x \in \mathbb{R}^{d_x}$, $x \sim \mathcal{N}(0, \Sigma_x)$, denote the state of a system. There are two agents indexed by $i \in \{1, 2\}$. Agent i, $i \in \{1, 2\}$, observes $y_i = C_i x + v_i$, $y_i \in \mathbb{R}^{d^i_y}$, where C_i is a $d^i_y \times d_x$ matrix and $v_i \in \mathbb{R}^{d^i_y}$, $v_i \sim \mathcal{N}(0, R_i)$. Assume that (x, v_1, v_2) are independent. The objective is for each agent to choose an estimate $\hat{z}_i \in \mathbb{R}^{d^i_z}$ according to $\hat{z}_i = g_i(y_i)$ (where g_i is a measurable function) to minimize

$$\mathbb{E}\left[\begin{bmatrix} L_1 x - \hat{z}_1 \\ L_2 x - \hat{z}_2 \end{bmatrix}^{\intercal} S \begin{bmatrix} L_1 x - \hat{z}_1 \\ L_2 x - \hat{z}_2 \end{bmatrix}\right],$$

where L_i and S are matrices of appropriate dimensions and S is positive definite.

Theorem 4 (in Appendix A) shows that the optimal estimates are given by

$$\hat{z}_i = F_i y_i,$$

where F_i is given by the solution of the following system of matrix equations:

$$\sum_{j \in \{1,2\}} \Big[S_{ij} F_j \Sigma_{ji} - S_{ij} L_j \Theta_i \Big] = 0, \quad \forall i \in \{1, 2\},$$

where $\Sigma_{ij} = \operatorname{cov}(y_i, y_j) = C_i \Sigma_x C_j^{\intercal}$ and $\Theta_i = \operatorname{cov}(x, y_i) = \Sigma_x C_i^{\intercal}$.

In contrast to the centralized case, the gains F_i ***depend on the weight matrix*** S. Thus, in team optimal decentralized state estimation, the weight matrix S plays an important role, which makes team optimal decentralized state estimation fundamentally different from centralized state estimation.

1.2. Contributions of the paper

We consider team optimal decentralized estimation of a linear stochastic process $\{x(t)\}_{t \geq 1}$, $x(t) \in \mathbb{R}^{d_x}$, by n-agents connected over a strongly connected communication graph with delays. The system has a non-classical information structure. Each agent observes a noisy version of the state and shares its observations with its neighbors. Agent i generates a state estimate $\hat{z}_i(t)$ based on all the information available to it (denoted by $I_i(t)$). Let $\hat{z}(t) = \operatorname{vec}(\hat{z}_1(t), \dots, \hat{z}_n(t))$ denote the estimate of all agents. The objective is to minimize the expected total estimation error where the per-step estimation error is $(Lx(t) - \hat{z}(t))^{\intercal} S (Lx(t) - \hat{z}(t))$ where L is an arbitrary matrix and S is a positive definite weight matrix.

Let L_i denote the i-th row of L. As shown in the previous section, the naive estimation strategy $\hat{z}_i(t) = L_i \mathbb{E}[x(t) | I_i(t)]$ does not minimize the weighted mean-square estimation error. Our main contribution is to systematically derive the optimal estimation strategy by combining ideas from team thoery [40, 42] and the common-information approach [41] with standard results in linear systems.

In particular, we split the information $I_i(t)$ at each agent into two parts: a common information $I^{\text{com}}(t)$ which is known to all agents and the remaining information at each agent which we call the local information and denote by $I^{\text{loc}}_i(t)$. Let

$$\hat{x}^{\text{com}}(t) = \mathbb{E}[x(t) | I^{\text{com}}(t)]$$

denote the common information based estimate of the current state and

$$\tilde{I}^{\text{loc}}_i(t) = I^{\text{loc}}_i(t) - \mathbb{E}[I^{\text{loc}}_i(t) | I^{\text{com}}(t)]$$

denote the "innovation" in the local observations with respect to the common information. Furthermore, let $\hat{\Sigma}_{ij}(t) = \operatorname{cov}(\tilde{I}^{\text{loc}}_i(t), \tilde{I}^{\text{loc}}_j(t))$ and $\hat{\Theta}_i(t) = \operatorname{cov}(x(t), \tilde{I}^{\text{loc}}_i(t))$.

Let $N = \{1, \dots, n\}$ denote the set of agents. Our main contributions is to show that the team optimal decentralized state estimate is given by

$$\hat{z}_i(t) = L_i(t) \hat{x}^{\text{com}}(t) + F_i(t) \tilde{I}^{\text{loc}}_i(t), \quad \forall i \in N,$$

where $\{F_i(t)\}_{i \in N}$ are gains that are computed by solving a linear system of matrix equations in terms of weights matrix S and the covariances $\{\hat{\Sigma}_{ij}(t)\}_{i,j \in N}$ and $\{\hat{\Theta}_i(t)\}_{i \in N}$. A salient feature of the result is that, in contrast to centralized state estimation, the gains $F_i(t)$ depend on the weight matrix S function of the weighted mean squared error.

We derive formulas to recursively compute the common information based state estimate $\hat{x}^{\text{com}}(t)$ and the covariances $\{\hat{\Sigma}_{ij}(t)\}_{i,j \in N}$ and $\{\hat{\Theta}_i(t)\}_{i \in N}$ from a single Kalman filter.

We prove that the results generalize to infinite horizon under the standard stabilizability and detectability conditions of the system matrices, a time homogeneous estimation strategy of the form

$$\bar{z}_i(t) = \bar{L}_i \hat{x}^{\text{com}}(t) + \bar{F}_i \tilde{I}^{\text{loc}}_i(t)$$

is optimal. Furthermore, the local estimation error covariances $\hat{\Sigma}_{ij}(t)$ and $\hat{\Theta}_{ij}(t)$ converge to time homogeneous limits $\bar{\Sigma}_{ij}$ and $\bar{\Theta}_{ij}$, which can be written in terms of the solution of an algebraic Riccati equation. Moreover, the gains $\{\bar{F}_i\}_{i \in N}$ can be computed by solving a linear system of matrix equations in terms of weight S and error covariance $\{\bar{\Sigma}_{ij}\}_{i,j} \in N$ and $\{\bar{\Theta}_i\}_{i \in N}$.

1.3. Notations

Given a matrix A, A_{ij} denotes its (i, j)-th element, $A_{i\bullet}$ denotes its i-th row, $A_{\bullet j}$ denotes its j-th column, $A^{\intercal}$ denotes its transpose, $\operatorname{vec}(A)$ denotes the column vector of A formed by vertically stacking the columns of A. Given a vector x, $\|x\|^2$ denotes $x^{\intercal} x$. Given matrices A and B, $\operatorname{diag}(A, B)$ denotes the matrix obtained by putting A and

B in diagonal blocks. Given matrices A and B with the same number of columns, $\mathrm{rows}(A, B)$ denotes the matrix obtained by stacking A on top of B. Given a square matrix A, $\mathrm{Tr}(A)$ denotes the sum of its diagonal elements. Given a symmetric matrix A, the notation $A > 0$ and $A \geq 0$ mean that A is positive definite and semi-definite, respectively. $\mathbf{0}_n$ is a square $n \times n$ matrix with all elements being equal to zero. $\mathbf{I}_n$ is the $n \times n$ identity matrix. We omit the subscript from $\mathbf{0}_n$ and $\mathbf{I}_n$ when the dimension is clear from context. We sometimes consider random vectors $X = (x_1, \dots, x_k)$ as a set with random elements $\{x_1, \dots, x_k\}$. In particular, given two random vectors $X = (x_1, \dots, x_k)$ and $Y = (y_1, \dots, y_m)$, we define $X \bigcap Y$ to mean $\mathrm{vec}(\{x_1, \dots, x_k\}) \bigcap \mathrm{vec}(\{y_1, \dots, y_m\})$. Similarly, we use $X \setminus Y$ to mean $\mathrm{vec}(\{x_1, \dots, x_k\}) \setminus \mathrm{vec}(\{y_1, \dots, y_m\})$.

Given any vector valued process $\{y(t)\}_{t \geq 1}$ and any time instances t_1, t_2 such that $t_1 \leq t_2$, $y(t_1{:}t_2)$ is a short hand notation for $\mathrm{vec}(y(t_1), y(t_1+1), \dots, y(t_2))$. Given matrices $\{A(i)\}_{i=1}^n$ with the same number of rows and vectors $\{w(i)\}_{i=1}^n$, $\mathrm{rows}(\bigodot_{i=1}^n A(i))$ and $\mathrm{vec}(\bigodot_{i=1}^n w(i))$ denote $\mathrm{rows}(A(1), \dots, A(n))$ and $\mathrm{vec}(w(1), \dots, w(n))$, respectively.

Given random vectors x and y, $\mathbb{E}[x]$ and $\mathrm{var}(x)$ denote the mean and variance of x while $\mathrm{cov}(x, y)$ denotes the covariance between x and y.

1.4. Preliminaries on Graph Theory

A directed weighted graph $\mathcal{G}$ is an ordered set (N, E, τ) where N is the set of nodes and $E \subset N \times N$ is the set of ordered edges, and $\tau\colon E \to \mathbb{R}^k$ is a weight function. An edge (i, j) in E is considered directed from i to j; i is the *in-neighbor* of j; j is the *out-neighbor* of i; and i and j are neighbors. The set of in-neighbors of i, called the *in-neighborhood* of i, is denoted by N_i^-; the set of out-neighbors of i, called the *out-neighborhood*, is denoted by N_i^+.

In a directed graph, a *directed path* $(v_1, v_2, \dots, v_k)$ is a weighted sequence of distinct nodes such that $(v_i, v_{i+1}) \in E$. The *length* of a path is the weighted number of edges in the path. The *geodesic distance* between two nodes i and j, denoted by ℓ_{ij}, is the shortest weight length of all paths connecting the two nodes. The weighted *diameter* of the graph is the largest weighted geodesic distance between any two nodes. A directed graph is called *strongly connected* if for every pair of nodes $i, j \in N$, there is a directed path from i to j and from j to i. A directed graph is called *complete* if for every pair of nodes $i, j \in N$, there is a directed edge from i to j and from j to i.

2. Problem Statement

2.1. Observation Model

Consider a linear stochastic process $\{x(t)\}_{t \geq 1}$, $x(t) \in \mathbb{R}^{d_x}$, where $x(1) \sim \mathcal{N}(0, \Sigma_x)$ and for $t \geq 1$,

$$x(t+1) = Ax(t) + w(t), \tag{1}$$

where A is a $d_x \times d_x$ matrix and $w(t) \in \mathbb{R}^{d_x}$, $w(t) \sim \mathcal{N}(0, Q)$, is the process noise. There are n agents, indexed by $N = \{1, \dots, n\}$, which observe the process with noise. At time t, the observation $y_i(t) \in \mathbb{R}^{d_y^i}$ of agent $i \in N$ is given by

$$y_i(t) = C_i x(t) + v_i(t), \tag{2}$$

where C_i is a $d_y^i \times d_x$ matrix and $v_i(t) \in \mathbb{R}^{d_y^i}$, $v_i(t) \sim \mathcal{N}(0, R_i)$, is the observation noise. Eq. (2) may be written in vector form as

$$y(t) = Cx(t) + v(t),$$

where $C = \mathrm{rows}(C_1, \dots, C_n)$, $y(t) = \mathrm{vec}(y_1(t), \dots, y_n(t))$, and $v(t) = \mathrm{vec}(v_1(t), \dots, v_n(t))$.

The agents are connected over a ***communication graph*** $\mathcal{G}$, which is a *strongly connected* weighted directed graph with vertex set N. For every edge (i, j), the associated weight τ_{ij} is a positive integer that denotes the communication delay from node i to node j.

Let $I_i(t)$ denote the information available to agent i at time t. We assume that agent i knows the history of all its observations and d_{ji} step delayed information of its in-neighbor j, $j \in N_i^-$, i.e.,

$$I_i(t) = \{y_i(1{:}t)\} \bigodot \Big(\bigodot_{j \in N_i^-} \{I_j(t - \tau_{ji})\} \Big). \tag{3}$$

In (3), we implicitly assume that $I_i(t) = \emptyset$ for any $t \leq 0$.

Let $\zeta_i(t) = I_i(t) \setminus I_i(t-1)$ denote the new information that becomes available to agent i at time t. Then, $\zeta_i(1) = y_i(1)$ and for $t > 1$,

$$I_i(t) = \mathrm{vec}(y_i(t), \{\zeta_j(t - d_{ji})\}_{j \in N_i^-}).$$

It is assumed that at each time t, agent $j, j \in N$, communicates $\zeta_j(t)$ to all its out-neighbors. This information reaches the out-neighbor i of agent j at time $t + d_{ji}$.

Some examples of the communication graph are as follows.

Example 1 Consider a complete graph with τ-step delay along each edge. The resulting information structure is

$$I_i(t) = \{y(1{:}t-\tau), y_i(t-\tau+1{:}t)\},$$

which is the *τ-step delayed sharing information structure* [43]. □

Example 2 Consider a strongly connected graph with unit delay along each edge. Let $\tau^* = \max_{i,j \in N} \ell_{ij}$, denote the weighted diameter of the graph and $N_i^k = \{j \in N : \ell_{ji} = k\}$ denote the k-hop in-neighbors of i with $N_i^0 = \{i\}$. The resulting information structure is

$$I_i(t) = \bigcup_{k=0}^{\tau^*} \bigcup_{j \in N_i^k} \{y_j(1{:}t-k)\},$$

which we call the *neighborhood sharing information structure*. □

At time t agent $i \in N$ generates an estimate $\hat{z}_i(t) \in \mathbb{R}^{d_z^i}$ of $L_i x(t)$ (where L_i is a $\mathbb{R}^{d_z^i \times d_x}$ matrix) according to

$$\hat{z}_i(t) = g_{i,t}(I_i(t)),$$

where $g_{i,t}$ is a measurable function called the *estimation rule* at time t. The collection $g_i \coloneqq (g_{i,1}, g_{i,2}, \dots)$ is called the *estimation strategy* of agent i and $g \coloneqq (g_1, \dots, g_n)$ is the *team estimation strategy profile* of all agents.

2.2. Estimation Cost

Let $\hat{z}(t) = \mathrm{vec}(\hat{z}_1(t), \dots, \hat{z}_n(t))$ denote the estimate of all agents. Then the estimation error $c(x(t), \hat{z}(t))$ is a weighted quadratic function of $(Lx(t) - \hat{z}(t))$. In particular,

$$c(x(t), \hat{z}(t)) = (Lx(t) - \hat{z}(t))^\intercal S (Lx(t) - \hat{z}(t)), \quad (4)$$

where S and L are defined as follows:

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}. \quad (5)$$

As an example of the cost function of the form (4), consider the following scenario. Suppose $x(t) = \mathrm{vec}(x_1(t), \dots, x_n(t))$, where we may think of $x_i(t)$ as the local state of agent $i \in N$. Suppose the agents want to estimate their own local state, but at the same time, want to make sure that the average $\bar{z}(t) \coloneqq \frac{1}{n}\sum_{i\in N} \hat{z}_i(t)$ of their estimates is close to the average $\bar{x}(t) \coloneqq \frac{1}{n}\sum_{i\in N} x_i(t)$ of their local states. In this case, the cost function is

$$c(x(t), \hat{z}(t)) = \sum_{i\in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2, \quad (6)$$

where $\lambda \in \mathbb{R}_{>0}$. This can be written in the form (4) with $L = \mathbf{I}$, and

$$S_{ij} = \begin{cases} (1 + \frac{\lambda}{n^2})\mathbf{I}, & i = j \\ \frac{\lambda}{n^2}\mathbf{I}, & i \neq j. \end{cases}$$

As an other example, suppose the agents are moving in a line (e.g.. a vehicular platoon) or in a closed shape (e.g., UAVs flying in a formation) and want to estimate their local state but, at the same time, want to ensure that the difference $\hat{d}_i(t) \coloneqq \hat{z}_i(t) - \hat{z}_{i+1}(t)$ between their estimates is close to the difference $d_i(t) \coloneqq x_i(t) - x_{i+1}(t)$ of their local states.

For example when agents are moving in a closed shape, the cost function is

$$c(x(t), \hat{z}(t)) = \sum_{i\in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \sum_{i\in N} \|d_i(t) - \hat{d}_i(t)\|^2, \quad (7)$$

where $\lambda \in \mathbb{R}_{>0}$. This can be written in the form (4) with $L = \mathbf{I}$ and

$$S_{ij} = \begin{cases} (1 + 2\lambda)\mathbf{I}, & i = j \\ -\lambda \mathbf{I}, & j \in \{i+1, i-1\} \pmod N \\ 0, & \text{otherwise}, \end{cases}$$

A similar weight matrix can be obtained for the case when agents are moving in a line.

2.3. Problem Formulation

We consider the following assumptions on the model.

(A1) The cost matrix S is positive definite.
(A2) The noise covariance matrices $\{R_i\}_{i\in N}$ are positive definite and Q and Σ_x are positive semi-definite.
(A3) The primitive random variables $(x(1), \{w(t)\}_{t\geq 1}, \{v_1(t)\}_{t\geq 1}, \dots, \{v_n(t)\}_{t\geq 1})$ are independent.
(A4) For any square root D of matrix Q such that $DD = Q$, (A, D) is stabilizable.
(A5) (A, C) is detectable.

We are interested in the following optimization problem.

Problem 1 (Finite Horizon) Given matrices A, $\{C_i\}_{i\in N}$, Σ_x, Q, $\{R_i\}_{i\in N}$, L, S, a *communication graph* $\mathcal{G}$ (and the corresponding weights d_{ij}), and a horizon T, choose a team estimation strategy profile g to minimize $J_T(g)$ given by

$$J_T(g) = \mathbb{E}^g \Big[\sum_{t=1}^{T} c(x(t), \hat{z}(t)) \Big]. \quad (8)$$

Problem 2 (Infinite Horizon) Given matrices A, $\{C_i\}_{i\in N}$, Σ_x, Q, $\{R_i\}_{i\in N}$, and a *communication graph* $\mathcal{G}$ (and the corresponding weights d_{ij}), choose a team estimation strategy profile g to minimize $\bar{J}(g)$ given by

$$\bar{J}(g) = \limsup_{T\to\infty} \frac{1}{T} J_T(g). \quad (9)$$

3. Main Results

3.1. Preliminaries on centralized Kalman filtering

Consider the centralized agent that observes $y(1{:}t-1)$ to generate an estimate $\hat{z}^{\mathrm{cen}}(t)$ to minimize

$$\mathbb{E}[(Lx(t) - \hat{z}^{\mathrm{cen}}(t))^\intercal (Lx(t) - \hat{z}^{\mathrm{cen}}(t))].$$

Again from [44],

$$\hat{z}^{\mathrm{cen}}(t) = L\hat{x}(t),$$

where $\hat{x}(t) = \mathbb{E}[x(t) | y(1{:}t-1)]$ is the delayed centralized estimate of the state. We have that $\hat{x}(1) = 0$ and for $t \geq 1$,

$$\hat{x}(t+1) = A\hat{x}(t) + AK(t)[y(t) - C\hat{x}(t)], \quad (10)$$

where

$$K(t) = P(t)C^\intercal [CP(t)C^\intercal + R]^{-1}, \quad (11)$$

and $P(t) = \mathrm{var}(x(t) - \hat{x}(t))$ is the covariance of the error $\tilde{x}(t) \coloneqq x(t) - \hat{x}(t)$. $P(t)$ can be pre-computed recursively using the forward Riccati equation: $P(1) = \Sigma_x$ and for $t \geq 1$,

$$P(t+1) = A\Delta(t)P(t)\Delta(t)^\intercal A^\intercal + AK(t)RK(t)^\intercal A^\intercal + Q, \quad (12)$$

where $\Delta(t) = I - K(t)C$.

3.2. Common information approach for decentralized estimation

Following [41], we define

$$I^{\mathrm{com}}(t) = \bigcap_{i \in N} I_i(t)$$

as the *common information* among all agents[1]. Since the information is shared over a strongly connected graph, the common information is

$$I^{\mathrm{com}}(t) = y(1{:}t - \tau^*),$$

where τ^* is the weighted diameter of the graph.

We define the local information at agent i as

$$I_i^{\mathrm{loc}}(t) = I_i(t) \setminus I^{\mathrm{com}}(t).$$

Then, $I_i(t) = I^{\mathrm{com}}(t) \cup I_i^{\mathrm{loc}}(t)$.

Furthermore, we define

$$\hat{x}^{\mathrm{com}}(t) = \mathbb{E}[x(t) | I^{\mathrm{com}}(t)]$$

as the *common estimate* of the state and

$$\hat{I}_i^{\mathrm{loc}}(t) = \mathbb{E}[I_i^{\mathrm{loc}}(t) | I^{\mathrm{com}}(t)]$$

as the common estimate of local information of agent i. Here we assume that $I_i^{\mathrm{loc}}(t)$ (and hence $\hat{I}_i^{\mathrm{loc}}(t)$) is a vector. Following [45], we define

$$\tilde{I}_i^{\mathrm{loc}}(t) = I_i^{\mathrm{loc}}(t) - \mathbb{E}[I_i^{\mathrm{loc}}(t) | I^{\mathrm{com}}(t)] \tag{13}$$

as the estimation error of the local information at agent i. We view $\tilde{I}_i^{\mathrm{loc}}(t)$ as the innovation in the local information given the common information.

To find a convenient expression for the *innovation term* $\tilde{I}_i^{\mathrm{loc}}(t)$, we follow [43] and express $I_i^{\mathrm{loc}}(t)$ in terms of the *delayed state* $x(t - \tau^* + 1)$. For that matter, for any $t, \ell \in \mathbb{Z}_{>0}$, define the $d_x \times 1$ random vector $w^{(k)}(\ell, t)$ as follows:

$$w^{(k)}(\ell, t) = \sum_{s = \max\{1, t-k\}}^{t-\ell-1} A^{t-\ell-s-1} w(s). \tag{14}$$

Note that $w^{(k)}(\ell, t) = 0$ if $t \le \min\{k, \ell + 1\}$ or $\ell \ge k$. For any $t \ge k$, we may write

$$x(t) = A^k x(t-k) + w^{(k)}(0, t), \tag{15}$$

$$y_i(t) = C_i A^k x(t-k) + C_i w^{(k)}(0, t) + v_i(t). \tag{16}$$

By definition $I_i^{\mathrm{loc}}(t) \subseteq y(t - \tau^* + 1{:}t)$. Thus, for any $i \in N$, we can identify matrix C_i^{loc} and random vectors $w_i^{\mathrm{loc}}(t)$ and $v_i^{\mathrm{loc}}(t)$ (which are linear functions of $w(t - \tau^* + 1{:}t - 1)$ and $v_i(t - \tau^* + 1{:}t)$) such that

$$I_i^{\mathrm{loc}}(t) = C_i^{\mathrm{loc}} x(t - \tau^* + 1) + w_i^{\mathrm{loc}}(t) + v_i^{\mathrm{loc}}(t). \tag{17}$$

To write the expressions for $(C_i^{\mathrm{loc}}, w_i^{\mathrm{loc}}(t), v_i^{\mathrm{loc}}(t))$ for the delayed sharing and neighborhood sharing information structures below, we define for any $\ell \le \tau^*$,

$$\mathcal{W}_i(\ell, t) = \begin{bmatrix} C_i w^{(\tau^*-1)}(\tau^* - 1, t) \\ C_i w^{(\tau^*-1)}(\tau^* - 2, t) \\ \vdots \\ C_i w^{(\tau^*-1)}(\ell, t) \end{bmatrix},$$

$$\mathcal{C}_i(\ell) = \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{\tau^* - \ell - 1} \end{bmatrix}, \quad \mathcal{V}_i(\ell, t) = \begin{bmatrix} v_i(t - \tau^* + 1) \\ v_i(t - \tau^* + 2) \\ \vdots \\ v_i(t - \ell) \end{bmatrix}. \quad \square$$

Example 1 (cont.) For the τ-step delayed sharing information structure $I_i^{\mathrm{loc}}(t) = y_i(t - \tau^* + 1{:}t)$. Thus,

$$C_i^{\mathrm{loc}} = \mathcal{C}_i(0), \quad w_i^{\mathrm{loc}}(t) = \mathcal{W}_i(0, t), \quad v_i^{\mathrm{loc}}(t) = \mathcal{V}_i(0, t).$$

Example 2 (cont.) For the neighborhood sharing information structure, $I_i(t) = \bigcup_{k=0}^{\tau^*} \bigcup_{j \in N_i^k} \{y_j(1{:}t - k)\}$. Thus,

$$C_i^{\mathrm{loc}} = \mathrm{rows}\Big(\bigodot_{\ell=0}^{\tau^*-1} \bigodot_{j \in N_i^\ell} \mathcal{C}_j(\ell) \Big),$$

$$w_i^{\mathrm{loc}}(t) = \mathrm{vec}\Big(\bigodot_{\ell=0}^{\tau^*-1} \bigodot_{j \in N_i^\ell} \mathcal{W}_j(\ell, t) \Big),$$

$$v_i^{\mathrm{loc}}(t) = \mathrm{vec}\Big(\bigodot_{\ell=0}^{\tau^*-1} \bigodot_{j \in N_i^\ell} \mathcal{V}_j(\ell, t) \Big). \quad \square$$

Now define,

$$\begin{aligned} \hat{x}(t - \tau^* + 1) &= \mathbb{E}[x(t - \tau^* + 1) \,|\, I^{\mathrm{com}}(t)] \\ &= \mathbb{E}[x(t - \tau^* + 1) \,|\, y(1{:}t - \tau^*)] \end{aligned} \tag{18}$$

as the *delayed centralized* estimate of the state and $\tilde{x}(t - \tau^* + 1) = x(t - \tau^* + 1) - \hat{x}(t - \tau^* + 1)$. Note that this notation is consistent with the notation for centralized Kalman filtering used in Section 3.1. Thus, $\hat{x}(t - \tau^* + 1)$ can be updated recursively using (10).

Lemma 1 *$w_i^{\mathrm{loc}}(t)$, $v_i^{\mathrm{loc}}(t)$, $\tilde{x}(t - \tau^* + 1)$, and $I^{\mathrm{com}}(t)$ are independent.* $\square$

PROOF Observe that $I^{\mathrm{com}}(t) = y(1{:}t - \tau^*)$ and $\tilde{x}(t - \tau^* + 1)$ are functions of the primitive random variables up to time $t - \tau^*$, while $w_i^{\mathrm{loc}}(t)$ and $v_i^{\mathrm{loc}}(t)$ are functions of the primitive random variables from time $t - \tau^* + 1$ onwards. Thus, $w_i^{\mathrm{loc}}(t)$ and $v_i^{\mathrm{loc}}(t)$ are independent of $\tilde{x}(t - \tau^* + 1)$ and $I^{\mathrm{com}}(t)$. Furthermore, (A3) implies that

[1] Our methodology relies on the split of the total information into common and local information as proposed in [41]. However, the specific details on how the common information is used is different from [41].

$w_i^{\text{loc}}(t)$ and $v_i^{\text{loc}}(t)$ are independent by assumption. Finally, note that $\tilde{x}(t-\tau^*+1)$ is the estimation error when estimating $x(t-\tau^*+1)$ given $I^{\text{com}}(t)$ and is, therefore, uncorrelated with $I^{\text{com}}(t)$. Since, all random variables are Gaussian, $\tilde{x}(t-\tau^*+1)$ and $I^{\text{com}}(t)$ being uncorrelated also means that they are independent. ■

From Lemma 1 and from (17), we get

$$\hat{I}_i^{\text{loc}}(t) = C_i^{\text{loc}}\hat{x}(t-\tau^*+1), \tag{19}$$

Our main result is as follows:

Theorem 1 *Under* (A1)–(A3)*, we have the following:*

1. *Optimal decentralized estimates are*

$$\hat{z}_i(t) = L_i\hat{x}^{\text{com}}(t) + F_i(t)\tilde{I}_i^{\text{loc}}(t), \tag{20}$$

where

$$\hat{x}^{\text{com}}(t) = A^{\tau^*-1}\hat{x}(t-\tau^*+1), \tag{21}$$

$\hat{x}(t-\tau^*+1)$ *is computed according to the centralized Kalman filtering* (10)–(12)*, and*

$$\begin{aligned}\tilde{I}_i^{\text{loc}}(t) &= I_i^{\text{loc}}(t) - C_i^{\text{loc}}\hat{x}(t-\tau^*+1)\\ &= C_i^{\text{loc}}\tilde{x}(t-\tau^*+1) + w_i^{\text{loc}}(t) + v_i^{\text{loc}}(t).\end{aligned} \tag{22}$$

2. *The optimal gains* $\{F_i(t)\}_{i\in N}$ *are given by the (unique) solution of the following system of matrix equations.*

$$\sum_{j\in N}\Big[S_{ij}F_j(t)\hat{\Sigma}_{ji}(t) - S_{ij}L_j\hat{\Theta}_i(t)\Big] = 0, \quad \forall i \in N, \tag{23}$$

where $\hat{\Sigma}_{ij}(t) = \operatorname{cov}\left(\tilde{I}_i^{\text{loc}}(t), \tilde{I}_j^{\text{loc}}(t)\right)$ *and is given by*

$$\hat{\Sigma}_{ij}(t) = C_i^{\text{loc}}P(t-\tau^*+1){C_j^{\text{loc}}}^{\intercal} + P_{ij}^w(t) + P_{ij}^v(t), \tag{24}$$

where $P_{ij}^w(t) = \operatorname{cov}(w_i^{\text{loc}}(t), w_j^{\text{loc}}(t))$, $P_{ij}^v(t) = \operatorname{cov}(v_i^{\text{loc}}(t), v_j^{\text{loc}}(t))$ *and* $\hat{\Theta}_i(t) = \operatorname{cov}(x(t), \tilde{I}_i^{\text{loc}}(t))$ *and is given by*

$$\hat{\Theta}_i(t) = [A^{\tau^*-1}P(t-\tau^*+1){C_i^{\text{loc}}}^{\intercal} + P_i^{\sigma}(t)], \tag{25}$$

where $P_i^{\sigma}(t) = \operatorname{cov}(w^{(\tau^*-1)}(0,t), w_i^{\text{loc}}(t))$*.*

3. *Finally, the optimal performance is given by*

$$J_T^* = \sum_{t=1}^{T}\Bigg[\operatorname{Tr}(L^{\intercal}SLP_0(t)) - \sum_{i\in N}\operatorname{Tr}\Bigg(F_i(t)^{\intercal}\sum_{j\in N}S_{ij}L_j\hat{\Theta}_i(t)\Bigg)\Bigg], \tag{26}$$

where $P_0(t) = \operatorname{var}(x(t) - \hat{x}^{\text{com}}(t))$ *and is given by*

$$P_0(t) = A^{\tau^*-1}P(t-\tau+1)(A^{\tau^*-1})^{\intercal} + \Sigma^w(t), \tag{27}$$

□

and $\Sigma^w(t) = \operatorname{var}(w^{(\tau^*-1)}(0,t))$*.*

Proof Since the choice of the estimates does not affect the evolution of the system, choosing an estimation profile $g = (g_1, \dots, g_n)$ to minimize $J_T(g)$ is equivalent to solving the following T separate optimization problems.

$$(P_t) \min_{(g_{1,t},\dots,g_{n,t})} \mathbb{E}[c(x(t), \hat{z}(t))], \quad \forall t \in \{1,\dots,T\}. \tag{28}$$

Problem (P_t) is a static problem with n agents. The information available at agent i is $I_i(t) = I^{\text{com}}(t) \cup I_i^{\text{loc}}(t)$ and the decision made by agent i is $\hat{z}_i(t)$. The per-step cost is $c(x(t), \hat{z}(t))$. Such a static team with common information is considered in the Appendix A and it is shown in Theorem 4 that the optimal decision is given by (20). We defer the proof of existence and uniqueness of the solution of (23) to Theorem 2.

The expression (21) for $\hat{x}^{\text{com}}(t)$ follows from (15). The expression (22) for $\tilde{I}_i^{\text{loc}}(t)$ follows from (17) and (19). Substituting (17) in (22), we get

$$\tilde{I}_i^{\text{loc}}(t) = C_i^{\text{loc}}\tilde{x}(t-\tau^*+1) + w_i^{\text{loc}}(t) + v_i^{\text{loc}}(t). \tag{29}$$

Thus, we get the expression (24) for $\hat{\Sigma}_{ij}(t)$ from Lemma 1.

From (15) and (29), and Lemma 1, we get the expression (25) for $\hat{\Theta}_i(t)$. Finally the expression for $P_0(t)$ follows from (15) and (21) and the performance of the strategy is given by (26). ■

Remark 1 Since we have assumed that the dynamics are time-homogeneous, the processes $\{w^{(\tau^*-1)}(0,t)\}_{t\ge\tau^*}$, $\{w_i^{\text{loc}}(t)\}_{t\ge\tau^*}$, and $\{v_i^{\text{loc}}(t)\}_{t\ge\tau^*}$ are stationary. Hence, for $t \ge \tau^*$, the covariance matrices $\Sigma^w(t)$, $P_{ij}^w(t)$, and $P_{ij}^v(t)$ are constant. □

Theorem 2 *Equation* (23) *has a unique solution and can be written more compactly as*

$$F(t) = \Gamma(t)^{-1}\eta(t), \tag{30}$$

where

$$\begin{aligned}F(t) &= \operatorname{vec}(F_1(t),\dots,F_n(t)),\\ \eta(t) &= \operatorname{vec}(S_{1\bullet}L\hat{\Theta}_1(t),\dots,S_{n\bullet}L\hat{\Theta}_n(t)),\\ \Gamma(t) &= [\Gamma_{ij}(t)]_{i,j\in N}, \quad \textit{where } \Gamma_{ij}(t) = \hat{\Sigma}_{ij}(t)\otimes S_{ij}.\end{aligned}$$

Furthermore, the optimal performance can be written as

$$J_T^* = \sum_{t=1}^{T}\big[\operatorname{Tr}(L^{\intercal}SLP_0(t)) - \eta(t)^{\intercal}\Gamma(t)^{-1}\eta(t)\big]. \tag{31}$$

□

Proof First, we start by observing that $\hat{\Sigma}_{ii}(t) > 0$. This follows from the fact that $\hat{\Sigma}_{ii}(t)$ is the variance of the innovation in the standard Kalman filtering equation. Thus, the positive definiteness of R_i in assumption (A2) ensures that $\hat{\Sigma}_{ii}(t)$ is positive definite [44, Section 3.4]. The result then follows from Lemma 7 in the Appendix. ■

Remark 2 In (20), the first term of the estimate is the conditional mean of the current state given the common information. The second term may be viewed as a "correction" which depends on the "innovation" in the local observations. A salient feature of the result is that the gains $\{F_i(t)\}_{i\in N}$ depend on the weight matrix S. □

When S is block diagonal, there is no cost coupling among the agents and Problem 1 reduces to n separate problems. Thus, the optimal estimates are $L_i\hat{x}_i(t)$. This can also be established from Theorem 1 as follows.

Corollary 1 *If $S_{ij} = 0$ for all $i, j \in N$, $i \neq j$, then*

$$\hat{z}_i(t) = L_i\hat{x}_i(t).$$

□

Proof For a block diagonal S, Eq. (23) reduces to

$$S_{ii}F_i(t)\hat{\Sigma}_{ii}(t) = S_{ii}L_i\hat{\Theta}_i(t). \tag{32}$$

Note that when S is block diagonal, (A3) implies that each S_{ii} is positive-definite, and hence invertible. Moreover, $\hat{\Sigma}_{ii}(t)$ is positive definite and invertible [44, Section 3.4]. Thus, Eq. (32) simplifies to $F_i(t) = L_i\hat{\Theta}_i(t)\hat{\Sigma}_{ii}^{-1}(t)$. Substituting this in (20) gives

$$\begin{aligned}\hat{z}_i(t) &= L_i\hat{x}^{\text{com}}(t) + L_i\hat{\Theta}_i(t)\hat{\Sigma}_{ii}^{-1}(t)\tilde{I}_i^{\text{loc}}(t)\\ &\overset{(a)}{=} L_i\big[\mathbb{E}[x(t)|I^{\text{com}}(t)] + \mathbb{E}[x(t)|\tilde{I}_i^{\text{loc}}(t)]\big]\\ &\overset{(b)}{=} L_i\big[\mathbb{E}[x(t)|I_i(t)],\end{aligned}$$

where (a) uses $\hat{x}^{\text{com}}(t) = \mathbb{E}[x(t)|I^{\text{com}}(t)]$ and the following equation for Gaussian zero mean random variables a and b:

$$\mathbb{E}[a|b] = \text{cov}(a, b)\,\text{var}(b)^{-1}b,$$

and (b) uses the orthogonal projection because $\tilde{I}_i^{\text{loc}}(t)$ is orthogonal to $I^{\text{com}}(t)$. ■

3.3. Comparison with [38, 39]

A decentralized estimation with a cost function similar to (4) but with $L_i = I$ was considered in [38, 39] under the assumption that there is no inter-agent communication. The solution methodology of [38, 39] was to lift the state and observation into $n(d_x)^2 \times nd_x$ and $\sum_{i\in N}(d_y^i)^2 \times \sum_{i\in N} d_y^i$ -dimensional matrices, respectively; use standard Kalman filtering to obtain a $n(d_x)^2 \times nd_x$ dimensional estimate and then use a linear transformation to obtain $\hat{z}(t)$. In contrast, we consider a model with inter-agent communication and obtain a solution that does not involve lifting the state to higher dimensions. In particular, our solution involves a d_x-dimensional Kalman filter instead of a $n^2(d_x)^3$ dimensional Kalman filer obtained in [38, 39].

3.4. Generalization to Infinite Horizon

We first state a standard result from centralized Kalman filtering [44].

Lemma 2 *Under (A2)–(A5), for any initial covariance $\Sigma_x \geq 0$, the sequence $\{P(t)\}_{t\geq 1}$ given by* (12) *is weakly increasing and bounded (in the sense of positive semi-definiteness). Thus it has a limit, which we denote by $\bar{P}$. Furthermore,*

1. *$\bar{P}$ does not depend on Σ_x.*
2. *$\bar{P}$ is positive semi-definite.*
3. *$\bar{P}$ is the unique solution to the following algebraic Riccati equation.*

$$\bar{P} = A\Delta\bar{P}\Delta^{\intercal}A^{\intercal} + A\bar{K}R\bar{K}^{\intercal}A^{\intercal} + Q, \tag{33}$$

□

where $\bar{K} = \bar{P}C^{\intercal}\big[C\bar{P}C^{\intercal} + R\big]^{-1}$ and $\Delta = I - \bar{K}C$.

4. *The matrix $(A - \bar{K}C)$ is asymptotically stable.*

Recall from Remark 2 that $\Sigma^w(t)$, $P_i^\sigma(t)$, $P_{ij}^w(t)$ and $P_{ij}^v(t)$ are constants for $t \geq \tau^*$. We denote the corresponding values for $t \geq \tau^*$ as $\bar{\Sigma}^w$, $\bar{P}_i^\sigma$, $\bar{P}_{ij}^w$, and $\bar{P}_{ij}^v$. Now define:

$$\begin{aligned}\bar{P}_0 &= A^{\tau^*-1}\bar{P}(A^{\tau^*-1})^{\intercal} + \bar{\Sigma}^w,\\ \bar{\Sigma}_{ij} &= C_i^{\text{loc}}\bar{P}C_j^{\text{loc}\intercal} + \bar{P}_{ij}^w + \bar{P}_{ij}^v,\end{aligned}$$

Lemma 3 *Under (A2)–(A5), we have the following:*

1. $\lim_{t\to\infty} P_0(t) = \bar{P}_0$.
2. $\lim_{t\to\infty} \hat{\Sigma}_{ij}(t) = \bar{\Sigma}_{ij}$.
3. $\lim_{t\to\infty} \hat{\Theta}_i(t) = \bar{\Theta}_i(t)$.

Proof Both relations follows immediately from Lemma 2 and Remark 1. ■

Theorem 3 *Under (A1)–(A5), the following estimation strategy is optimal for Problem 2:*

$$\hat{z}_i(t) = L_i\hat{x}^{\text{com}}(t) + \bar{F}_i\tilde{I}_i^{\text{loc}}(t), \tag{34}$$

where the gains $\{\bar{F}_i\}_{i\in N}$ satisfy the following system of matrix equations:

$$\sum_{j\in N}\Big[S_{ij}\bar{F}_j\bar{\Sigma}_{ji} - S_{ij}L_j\bar{\Theta}_i\Big] = 0, \quad \forall i \in N. \tag{35}$$

Eq. (35) has a unique solution and can be written more compactly as

$$\bar{F} = \bar{\Gamma}^{-1}\bar{\eta}, \tag{36}$$

where

$$\begin{aligned}\bar{F} &= \text{vec}(\bar{F}_1, \dots, \bar{F}_n),\\ \bar{\eta} &= \text{vec}(S_{1\bullet}L\bar{\Theta}_1, \dots, S_{n\bullet}L\bar{\Theta}_n),\\ \bar{\Gamma}(t) &= [\bar{\Gamma}_{ij}]_{i,j\in N}, \quad \textit{where } \bar{\Gamma}_{ij} = \bar{\Sigma}_{ij}\otimes S_{ij}.\end{aligned}$$

Furthermore, the optimal performance is given by

$$J^* = \text{Tr}(L^{\intercal}SL\bar{P}_0) - \bar{\eta}^{\intercal}\bar{\Gamma}^{-1}\bar{\eta}. \tag{37}$$

□

Proof $\bar{\Sigma}_{ii}$ is the variance of the innovation in the standard Kalman filtering equation and by positive definiteness of R_i is positive definite. Lemma 7 in the Appendix implies that (35) has a unique solution that is given by (36). To show the strategy (34) is optimal, we proceed in two steps. We first identify a lower bound in optimal performance and then show that the proposed strategy achieves that lower bound.

Step 1. From Theorem 1, for any strategy g, we have that

$$\frac{1}{T} J_T(g) \geq \frac{1}{T}\sum_{t=1}^{T}\left[\operatorname{Tr}(L^\intercal S L P_0(t)) - \eta(t)^\intercal \Gamma(t)\eta(t)\right]$$

Taking limits of both sides and using Lemma 3 (which implies that $\lim_{t\to\infty}\eta(t) = \bar{\eta}$ and $\lim_{t\to\infty}\Gamma(t) = \bar{\Gamma}$), we get

$$\limsup_{T\to\infty}\frac{1}{T}J_T(g) \geq \operatorname{Tr}(L^\intercal S L \bar{P}_0) - \bar{\eta}^\intercal\bar{\Gamma}\bar{\eta} = J^* \tag{38}$$

Step 2. Suppose $\hat{z}(t)$ is chosen according to strategy (36) and let $J(t)$ denote $\mathbb{E}[c(x(t),\hat{z}(t))]$. Following (A.16) and (A.17) in the proof of Theorem 1, we have that

$$\begin{aligned} J(t) &= \operatorname{Tr}(L^\intercal S L P_0(t)) \\ &\quad - \sum_{i\in N}\operatorname{Tr}\left(\bar{F}_i^\intercal \sum_{j\in N}\left[2S_{ij}L_j\hat{\Theta}_i(t) - S_{ij}\bar{F}_j\hat{\Sigma}_{ji}(t)\right]\right). \end{aligned}$$

From Lemma 3, we have that

$$\begin{aligned} \lim_{t\to\infty} J(t) &= \operatorname{Tr}(L^\intercal S L\bar{P}_0) \\ &\quad - \sum_{i\in N}\operatorname{Tr}\left(\bar{F}_i^\intercal\sum_{j\in N}\left[2S_{ij}L_j\bar{\Theta}_i - S_{ij}\bar{F}_j\bar{\Sigma}_{ji}\right]\right). \\ &= \operatorname{Tr}(L^\intercal S L\bar{P}_0) - \bar{\eta}^\intercal\bar{\Gamma}\bar{\eta} = J^*. \end{aligned}$$

Thus, by Cesaro's mean theorem, we get $\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}J(t) = J^*$. Hence, the strategy (36) achieves the lower bound of (38) and is therefore optimal. ∎

4. Discussion of the results

4.1. System model

Consider a 4-dimensional stochastic process $\{x(t)\}_{t\geq 1}$, $x(t)\in\mathbb{R}^4$, where $x(1)\sim\mathcal{N}(0,I)$ and

$$x(t+1) = Ax(t) + w(t),$$

where $w(t) = \mathcal{N}(0,Q)$.

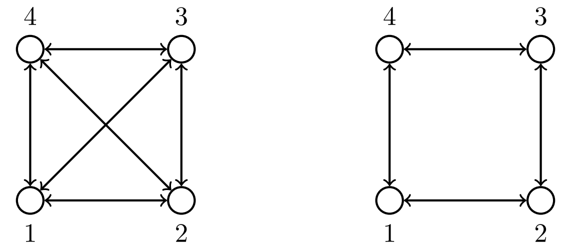
The process is observed by 4 agents, where agent i observes component i of the state with noise, i.e.,

$$y_i(t) = x_i(t) + v_i(t),$$

where $v_i(t)\sim\mathcal{N}(0,R_i)$. Note that

$$C = \operatorname{rows}(C_1, C_2, C_3, C_4) = I.$$

We consider two information structures. For both information structures, we assume that the cost function is given by both (6) and (7).



(a) A system with 2-step delay sharing information structure.

(b) A system with neighborhood sharing information structure.

Figure 1: Two systems with different information structures.

4.1.1. (IS-1) *Information structure 1*

Complete graph with 2-step delayed information structure, shown in Fig. 1a. The information structure is given by

$$I_i(t) = \{y(1{:}t-2), y_i(t-1{:}t)\}.$$

4.1.2. (IS-2) *Information structure 2*

Neighborhood sharing information structure shown in Fig. 1b. The information structure is given by

$$I_i(t) = \{y(1{:}t-2), y_{i-1}(t-1), y_i(t-1{:}t), y_{i+1}(t-1)\},$$

where we have assumed that the subscripts $i+1$ and $i-1$ are evaluated modulo 4 over the residue system $\{1,2,3,4\}$.

4.2. Computation of intermediate variable

For both information structures, we show the computations for C_i^{loc}, $w_i^{\text{loc}}(t)$, $v_i^{\text{loc}}(t)$, $P_i^\sigma(t)$, $P_{ij}^w(t)$, and $P_{ij}^v(t)$.

4.2.1. Information structure 1 (IS-1)

$I^{\text{com}}(t) = y(1{:}t-2)$ and $I_i^{\text{loc}}(t) = y_i(t-1{:}t)$ and

$$C_i^{\text{loc}} = \begin{bmatrix} C_i \\ C_iA\end{bmatrix} = \begin{bmatrix} C_i \\ A_{i\bullet}\end{bmatrix}, v_i^{\text{loc}}(t) = \begin{bmatrix} v_i(t-1) \\ v_i(t)\end{bmatrix},$$

$$w_i^{\text{loc}}(t) = \begin{bmatrix} 0 \\ C_iw(t-1)\end{bmatrix} = \begin{bmatrix} 0 \\ w_i(t-1)\end{bmatrix}.$$

Using these, we get that

- For $t = 1$,

$$\begin{aligned} \Sigma^w(1) &= 0, & P_i^\sigma(1) &= \begin{bmatrix}0 & 0\end{bmatrix}, \\ P_{ii}^w(1) &= \operatorname{diag}(0,0), & P_{ij}^w(1) &= \operatorname{diag}(0,0), \\ P_{ii}^v(1) &= \operatorname{diag}(0,R_i), & P_{ij}^v(1) &= \operatorname{diag}(0,0). \end{aligned}$$

- For $t \geq 2$,

$$\begin{aligned} \Sigma^w(t) &= Q, \\ P_i^\sigma(t) &= \begin{bmatrix}0 & QC_i^\intercal\end{bmatrix} = \begin{bmatrix}0 & Q_{\bullet i}\end{bmatrix}, \\ P_{ij}^w(t) &= \operatorname{diag}(0, C_iQC_j^\intercal) = \operatorname{diag}(0, Q_{ij}), \\ P_{ii}^v(t) &= \operatorname{diag}(R_i, R_i), \\ P_{ij}^v(t) &= \operatorname{diag}(0,0). \end{aligned}$$

Substituting these, we get that $\hat{\Sigma}_{ij}(1) = \delta_{ij}\mathrm{diag}(0, R_i)$, and for $t \geq 2$,

$$\hat{\Sigma}_{ij}(t) = \begin{bmatrix} C_i \\ C_i A \end{bmatrix} P(t-1) \begin{bmatrix} C_j \\ C_j A \end{bmatrix}^{\intercal} + \begin{bmatrix} \delta_{ij} R_i & 0 \\ 0 & C_i Q C_j^{\intercal} + \delta_{ij} R_i \end{bmatrix}$$
$$= \begin{bmatrix} C_i \\ A_{i\bullet} \end{bmatrix} P(t-1) \begin{bmatrix} C_j \\ A_{j\bullet} \end{bmatrix}^{\intercal} + \begin{bmatrix} \delta_{ij} R_i & 0 \\ 0 & Q_{ij} + \delta_{ij} R_i \end{bmatrix}.$$

Finally, substituting $\hat{\Sigma}_{ij}(t)$ in (23) or (30) gives us the optimal gains.

4.2.2. Information structure 2 (IS-2)

$I^{\mathrm{com}}(t) = y(1{:}t-2)$ and

$$I_i^{\mathrm{loc}}(t) = \{y_{i-1}(t-1), y_i(t-1{:}t), y_{i+1}(t-1)\}.$$

Thus,

$$C_i^{\mathrm{loc}} = \mathrm{rows}(C_{i-1}, C_i, C_i A, C_{i+1}) = \mathrm{rows}(C_{i-1}, C_i, A_{i\bullet}, C_{i+1}),$$

and

$$w_i^{\mathrm{loc}}(t) = \begin{bmatrix} 0 \\ 0 \\ C_i w(t-1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w_i(t-1) \\ 0 \end{bmatrix}, \quad v_i^{\mathrm{loc}}(t) = \begin{bmatrix} v_{i-1}(t-1) \\ v_i(t-1) \\ v_i(t) \\ v_{i+1}(t-1) \end{bmatrix}.$$

Using these, we get that

- For $t = 1$,

$$\begin{aligned} &\Sigma^w(1) = 0, && P_i^{\sigma}(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \\ &P_{ii}^w(1) = \mathrm{diag}(0,0,0,0), && P_{ij}^w(1) = \mathrm{diag}(0,0,0,0), \\ &P_{ii}^v(1) = \mathrm{diag}(0,0,R_i,0), && P_{ij}^v(1) = \mathrm{diag}(0,0,0,0). \end{aligned}$$

- for $t \geq 2$,

$$\begin{aligned} &\Sigma^w(t) = Q, \\ &P_i^{\sigma}(t) = [0, 0, Q_{\bullet i}, 0], \\ &P_{ij}^w(t) = \mathrm{diag}(0, 0, C_i Q C_j^{\intercal}, 0) = \mathrm{diag}(0,0,Q_{ij},0) \\ &P_{ii}^v(t) = \mathrm{diag}(R_{i-1}, R_i, R_i, R_{i+1}), \end{aligned}$$

$$P_{i,i+1}^v(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & R_{i+1} & 0 & 0 \end{bmatrix}, P_{i,i+2}^v(t) = \begin{bmatrix} 0 & 0 & 0 & R_{i-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R_{i+1} & 0 & 0 & 0 \end{bmatrix},$$

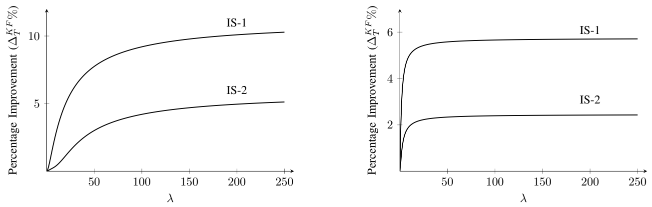
$$P_{i+1,i}^v(t) = P_{i,i+1}^v(t)^{\intercal}, \text{ and } P_{i+2,i}^v(t) = P_{i,i+2}^v(t)^{\intercal}.$$

4.3. Performance evaluation

For both information structures, we compute the performance of the optimal estimation strategy for the following choices of parameters:

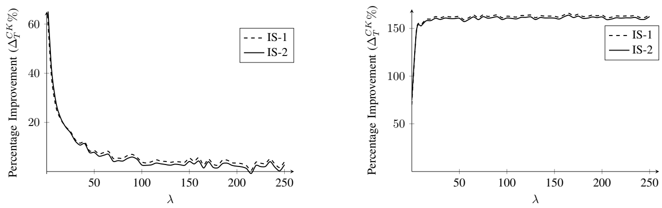
$$A = \begin{bmatrix} 0.7 & 0.1 & 0 & 0.2 \\ 0.2 & 0.6 & 0 & 0.3 \\ 0.1 & 0.2 & 0.7 & 0.5 \\ 0 & 0.2 & 0 & 0.7 \end{bmatrix},$$

$Q = I, R = 0.1I$, and $T = 100$.



(a) Per-step cost (6). (b) Per-step cost (7).

Figure 2: Percentage improvement over Kalman filtering.



(a) Per-step cost (6). (b) Per-step cost (7).

Figure 3: Percentage improvement over consensus Kalman filtering.

We compare the optimal performance with two baselines. The first is Kalman filtering where, each agent ignores the cost coupling and simply generates

$$\hat{z}_i^{\mathrm{KF}}(t) = L_i \mathbb{E}[x(t) | I_i(t)], \tag{39}$$

as its estimate. It can be shown that performance of the Kalman filtering strategy is

$$\begin{aligned} J_T^{\mathrm{KF}} &= \mathrm{Tr}(L^{\intercal} S L P_0(t)) \\ &+ \sum_{i \in N} \mathrm{Tr}\left(K_i(t)^{\intercal} \sum_{j \in N} S_{ij} \Big[K_j(t) \hat{\Sigma}_{ji}(t) - 2 L_j \hat{\Theta}_i(t) \Big] \right). \end{aligned} \tag{40}$$

The second is a consensus based Kalman filter as described in [29]. We don't have a closed form expression for the weighted mean square error of the consensus Kalman filter, so we evaluate the performance J_T^{CK} using Monte Carlo evaluation averaged over 1000 sample paths. We consider two types of per-step cost given in (6) and (7). To compare the performance of the strategy given in Theorem 1 with the two baselines, we define

$$\Delta_T^{\mathrm{KF}} = \frac{J_T^{\mathrm{KF}} - J_T^*}{J_T^*} \quad \text{and} \quad \Delta_T^{\mathrm{CK}} = \frac{J_T^{\mathrm{CK}} - J_T^*}{J_T^*},$$

as the relative improvement in performance of the optimal strategy compared to Kalman filtering and consensus Kalman filtering. The relative performance as a function of λ is shown in Fig. 2 and Fig. 3. These plots show that team optimal strategy may outperform naive Kalman filtering strategy by $5-10\%$. This improvement in performance will increase with the number of agents.

5. Conclusion

In this paper, we investigate the team optimal decentralized estimation of linear stochastic system by multiple agents. Each agent receives a noisy observation of the state of the process and agents share their observations over a strongly connected communication graph. Since the graph is strongly connected, all agents know the τ^*-step delayed observations (where τ^* is the weighted diameter of the graph), which we call the common information. We show that the optimal estimate is given by

$$\hat{z}_i(t) = L_i \hat{x}^{\text{com}}(t) + F_i(t)\tilde{I}_i^{\text{loc}}(t).$$

The first term of the estimate is the conditional mean of the current state given the common information. The second term may be viewed as a "correction" which depends on the "innovation" in the local observations. A salient feature of the result is that the gains $\{F_i(t)\}_{i\in N}$ depend on the weight matrix S.

We also show that $\hat{x}^{\text{com}}(t)$ and $\tilde{I}_i^{\text{loc}}(t)$ may be computed in terms of $\hat{x}(t-\tau^*+1)$, which is the conditional mean of the (τ^*-1)-step delayed state given the τ^*-step delayed observation (i.e., the common information). Note that $\hat{x}(t-\tau^*+1)$ can be recursively updated using standard Kalman Filtering equation.

In Appendix B, we consider decentralized control problem with one-step delayed information structure. We show that the decentralized control problem is equivalent to a team optimal decentralized estimation problem. And the lack of separation between estimation and control can be explained by the salient feature of decentralized estimation: the team optimal decentralized estimates depend on the weight matrix of the estimation cost. Nonetheless, as explained in Appendix B, the lack of separation does not impede the calculations of the optimal gains for the decentralized control problem with one-step delayed sharing. However, there are information structures [8, 12] for which the forward and backward Riccati equations are coupled. We hope that the decentralized estimation viewpoint, considered in this paper, may provide new insight for such models as well.

Acknowledgement

The authors are grateful to Peter Caines, Roland Malhame, and Demosthenis Teneketzis for useful discussion and feedback.

References

[1] H. S. Witsenhausen, A counterexample in stochastic optimum control, SIAM Journal on Control 6 (1) (1968) 131–147 (1968).

[2] P. Whittle, J. Rudge, The optimal linear solution of a symmetric team control problem, Journal of Applied Probability 11 (2) (1974) 377–381 (1974). doi:10.2307/3212760.

[3] Y.-C. Ho, K.-C. Chu, Team decision theory and information structures in optimal control problems–part I, IEEE Trans. Autom. Control 17 (1) (1972) 15–22 (Feb 1972).

[4] S. Yuksel, Stochastic nestedness and the belief sharing information pattern, IEEE Transactions on Automatic Control 54 (12) (2009) 2773–2786 (Dec 2009). doi:10.1109/TAC.2009.2031723.

[5] A. Lamperski, J. C. Doyle, Dynamic programming solutions for decentralized state-feedback LQG problems with communication delays, in: 2012 American Control Conference (ACC), 2012, pp. 6322–6327 (June 2012). doi:10.1109/ACC.2012.6315282.

[6] L. Lessard, S. Lall, Quadratic invariance is necessary and sufficient for convexity, in: Proceedings of the 2011 American Control Conference, 2011, pp. 5360–5362 (June 2011). doi:10.1109/ACC.2011.5990928.

[7] L. Lessard, S. Lall, Optimal control of two-player systems with output feedback, IEEE Trans. Autom. Control 60 (8) (2015) 2129–2144 (Aug 2015).

[8] L. Lessard, S. Lall, Optimal controller synthesis for the decentralized two-player problem with output feedback, in: 2012 American Control Conference (ACC), 2012, pp. 6314–6321 (June 2012).

[9] N. Nayyar, D. Kalathil, R. Jain, Optimal decentralized control in unidirectional one-step delayed sharing pattern with partial output feedback, in: 2014 American Control Conference, 2014, pp. 1906–1911 (June 2014).

[10] N. Sandell, M. Athans, Solution of some nonclassical LQG stochastic decision problems, IEEE Trans. Autom. Control 19 (2) (1974) 108–116 (Apr 1974).

[11] B. Z. Kurtaran, R. Sivan, Linear-quadratic-gaussian control with one-step-delay sharing pattern, IEEE Trans. Autom. Control 19 (5) (1974) 571–574 (Oct 1974).

[12] N. Nayyar, D. Kalathil, R. Jain, Optimal decentralized control with asymmetric one-step delayed information sharing, IEEE Trans. Control Netw. Syst. 5 (1) (2018) 653–663 (March 2018).

[13] J. Li, G. AlRegib, Distributed estimation in energy-constrained wireless sensor networks, IEEE Transactions on Signal Processing 57 (10) (2009) 3746–3758 (Oct 2009). doi:10.1109/TSP.2009.2022874.

[14] Y. F. Huang, S. Werner, J. Huang, N. Kashyap, V. Gupta, State estimation in electric power grids: Meeting new challenges presented by the requirements of the future grid, IEEE Signal Processing Magazine 29 (5) (2012) 33–43 (Sept 2012). doi:10.1109/MSP.2012.2187037.

[15] Y. Guo, W. Wu, B. Zhang, H. Sun, A distributed state estimation method for power systems incorporating linear and nonlinear models, International Journal of Electrical Power & Energy Systems 64 (2015) 608 – 616 (2015).

[16] D. Marelli, B. Ninness, M. Fu, Distributed weighted least-squares estimation for power networks, IFAC-PapersOnLine 48 (28) (2015) 562 – 567, 17th IFAC Symposium on System Identification SYSID 2015 (2015).

[17] F. Garin, L. Schenato, A Survey on Distributed Estimation and Control Applications Using Linear Consensus Algorithms, Springer London, London, 2010, pp. 75–107 (2010).

[18] S. S. Stankovic, M. J. Stanojevic, D. D. Siljak, Decentralized overlapping control of a platoon of vehicles, IEEE Transactions on Control Systems Technology 8 (5) (2000) 816–832 (Sep 2000). doi:10.1109/87.865854.

[19] H. R. Feyzmahdavian, A. Alam, A. Gattami, Optimal distributed controller design with communication delays: Application to vehicle formations, in: IEEE Conf. on Decision and Control (CDC), 2012, pp. 2232–2237 (2012).

[20] C. Sanders, E. Tacker, T. Linton, A new class of decentralized filters for interconnected systems, IEEE Transactions on Automatic Control 19 (3) (1974) 259–262 (Jun 1974). doi:10.1109/TAC.1974.1100541.

[21] C. Sanders, E. Tacker, T. Linton, R. Ling, Specific structures for large-scale state estimation algorithms having information exchange, IEEE Trans. Autom. Control 23 (2) (1978) 255–261 (Apr 1978). doi:10.1109/TAC.1978.1101715.

[22] J. Speyer, Computation and transmission requirements for a decentralized linear-quadratic-Gaussian control problem, IEEE Trans. Autom. Control 24 (2) (1979) 266–269 (April 1979). doi:10.1109/TAC.1979.1101973.

[23] C.-Y. Chong, Hierarchical estimation, in: Proc. MIT/ONR Workshop on C3, 1979 (1979).
[24] M. Hassan, G. Salut, M. Singh, A. Titli, A decentralized computational algorithm for the global Kalman filter, IEEE Trans. Autom. Control 23 (2) (1978) 262–268 (Apr 1978). doi:10.1109/TAC.1978.1101713.
[25] A. Willsky, M. Bello, D. Castanon, B. Levy, G. Verghese, Combining and updating of local estimates and regional maps along sets of one-dimensional tracks, IEEE Trans. Autom. Control 27 (4) (1982) 799–813 (Aug 1982). doi:10.1109/TAC.1982.1103019.
[26] Z. Duda, Fusion Kalman filtration for distributed multisensor systems, Archives of Control Sciences 24 (1) (2014) 53–65 (2014).
[27] Z. Duda, Fusion Kalman filtration with k-step delay sharing pattern, Archives of Control Sciences 25 (3) (2015) 307–318 (2015).
[28] D. Castanon, D. Teneketzis, Distributed estimation algorithms for nonlinear systems, IEEE Trans. Autom. Control 30 (5) (1985) 418–425 (May 1985). doi:10.1109/TAC.1985.1103972.
[29] R. Olfati-Saber, Distributed Kalman filtering for sensor networks, in: IEEE Conf. on Decision and Control (CDC), 2007, pp. 5492–5498 (Dec 2007). doi:10.1109/CDC.2007.4434303.
[30] R. Olfati-Saber, Kalman-consensus filter : Optimality, stability, and performance, in: Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference, 2009, pp. 7036–7042 (Dec 2009). doi:10.1109/CDC.2009.5399678.
[31] R. Olfati-Saber, P. Jalalkamali, Coupled distributed estimation and control for mobile sensor networks, IEEE Trans. Autom. Control 57 (10) (2012) 2609–2614 (Oct 2012). doi:10.1109/TAC.2012.2190184.
[32] G. Battistelli, L. Chisci, G. Mugnai, A. Farina, A. Graziano, Consensus-based linear and nonlinear filtering, IEEE Trans. Autom. Control 60 (5) (2015) 1410–1415 (May 2015). doi:10.1109/TAC.2014.2357135.
[33] F. S. Cattivelli, A. H. Sayed, Diffusion strategies for distributed Kalman filtering and smoothing, IEEE Trans. Autom. Control 55 (9) (2010) 2069–2084 (Sept 2010). doi:10.1109/TAC.2010.2042987.
[34] V. Borkar, P. Varaiya, Asymptotic agreement in distributed estimation, IEEE Trans. Autom. Control 27 (3) (1982) 650–655 (June 1982). doi:10.1109/TAC.1982.1102982.
[35] J. Tsitsiklis, M. Athans, Convergence and asymptotic agreement in distributed decision problems, IEEE Transactions on Automatic Control 29 (1) (1984) 42–50 (January 1984). doi:10.1109/TAC.1984.1103385.
[36] R. B. Washburn, D. Teneketzis, Asymptotic agreement among communicating decisionmakers, in: 1983 American Control Conference, 1983, pp. 1331–1336 (June 1983).
[37] D. Teneketzis, P. Varaiya, Consensus in distributed estimation with inconsistent beliefs, in: IEEE Conference on Decision and Control (CDC), 1984, pp. 1200–1200 (Dec 1984). doi:10.1109/CDC.1984.272207.
[38] S. M. Barta, On linear control of decentralized stochastic systems, Ph.D. thesis, Massachusetts Institute of Technology (1978).
[39] M. S. Andersland, D. Teneketzis, Measurement scheduling for recursive team estimation, Journal of Optimization Theory and Applications 89 (3) (1996) 615–636 (Jun 1996).
[40] R. Radner, Team decision problems, The Annals of Mathematical Statistics 33 (3) (1962) 857–881 (1962).
[41] A. Nayyar, A. Mahajan, D. Teneketzis, Decentralized stochastic control with partial history sharing: A common information approach, IEEE Trans. Autom. Control 58 (7) (2013) 1644–1658 (July 2013).
[42] J. Marschak, R. R, Economic theory of teams, Yale University Press, New Haven, 1972 (1972).
[43] H. S. Witsenhausen, Separation of estimation and control for discrete time systems, Proc. IEEE 59 (11) (1971) 1557–1566 (1971).
[44] P. E. Caines, Linear Stochastic Systems, John Wiley & Sons, Inc., New York, NY, USA, 1987 (1987).
[45] T. Kailath, An innovations approach to least-squares estimation–part I: Linear filtering in additive white noise, IEEE Trans. Autom. Control 13 (6) (1968) 646–655 (December 1968). doi:10.1109/TAC.1968.1099025.
[46] T. Yoshikawa, Dynamic programming approach to decentralized stochastic control problems, IEEE Trans. Autom. Control 20 (6) (1975) 796–797 (Dec 1975).
[47] K. J. Astrom, Introduction to stochastic control theory, Academic Press New York, 1970 (1970).
[48] P. R. Kumar, P. Varaiya, Stochastic Systems: Estimation, Identification and Adaptive Control, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1986 (1986).

Appendix A. One-step decentralized state estimation

In this appendix, we consider a one-step decentralized estimation problem with common observation between controllers and show that the structure of optimal estimators has a form similar to Theorem 1. The proof of Theorem 1 builds up on the result of this appendix.

Appendix A.1. System model and problem formulation

Consider a system that consists of n agents that are indexed by the set $N = \{1, \dots, n\}$. We use N_0 to denote the set $\{0, 1, \dots, n\}$.

Let $(x, y_0, y_1, \dots, y_n)$, where $x \in \mathbb{R}^{d_x}$ and $y_i \in \mathbb{R}^{\tau^i_y}$ for $i \in N_0$, be jointly Gaussian zero-mean random variables. For any $i, j \in N_0$, let $\Theta_i = \operatorname{cov}(x, y_i)$ and $\Sigma_{ij} = \operatorname{cov}(y_i, y_j)$.

Agent i, $i \in N$, observes (y_0, y_i) and chooses an estimate $\hat{z}_i \in \mathbb{R}^{d^i_z}$ according to an estimation rule g_i, i.e., $\hat{z}_i = g_i(y_0, y_i)$. The performance is measured by the estimation error given by:

$$c(x, \hat{z}_1, \dots, \hat{z}_n) = \sum_{i \in N} \sum_{j \in N} (L_i x - \hat{z}_i)^\intercal S_{ij} (L_j x - \hat{z}_j), \quad \text{(A.1)}$$

where $\{S_{ij}\}_{i,j} \in \mathbb{R}^{d^i_z \times d^j_z}$ and L is any matrix. For ease of notation, define $\hat{z} = \operatorname{vec}(\hat{z}_1, \dots, \hat{z}_n)$.

Then, the cost (B.6) may be written succinctly as

$$c(x, \hat{z}) = (Lx - \hat{z})^\intercal S (Lx - \hat{z}), \quad \text{(A.2)}$$

where S and L are given by (5). We assume that the weight matrix S is positive definite.

We are interested in the following optimization problem.

Problem 3 Given the covariance matrices $\{\Theta_i\}_{i \in N_0}$ and $\{\Sigma_{ij}\}_{i,j \in N_0}$ and weight matrices L and S, choose the estimation strategy $g = (g_1, \dots, g_n)$ to minimize the expected estimation error $J(g)$ given by

$$J(g) \coloneqq \mathbb{E}[c(x, \hat{z})]. \quad \text{(A.3)}$$

Appendix A.2. Structure of optimal estimation strategy

We define the *local estimate* as $\hat{x}_i = \mathbb{E}[x | y_i, y_0]$, the *common estimate* as $\hat{x}_0 = \mathbb{E}[x | y_0]$, $\hat{y}_i = \mathbb{E}[y_i | y_0]$, and $\tilde{y}_i = y_i - \hat{y}_i$. Let $\hat{\Theta}_i = \operatorname{cov}(x, y_i - \hat{y}_i)$ and $\hat{\Sigma}_{ij} = \operatorname{cov}(y_i - \hat{y}_i, y_j - \hat{y}_j)$. Then, we have the following.

Theorem 4 *The team optimal estimation strategy in Problem 3 is linear in the common estimates. Specifically, the optimal estimate may be written as*

$$\hat{z}_i = L_i \hat{x}_0 + F_i \tilde{y}_i, \quad \forall i \in N, \tag{A.4}$$

where the gains $\{F_i\}_{i\in N}$ satisfy the following system of matrix equations:

$$\sum_{j\in N} \left[S_{ij} F_j \hat{\Sigma}_{ji} - S_{ij} L_j \hat{\Theta}_i \right] = 0, \quad \forall i \in N. \tag{A.5}$$

If $\hat{\Sigma}_{ii} > 0$ for all $i \in N$, then (A.5) *has a unique solution and can be written as*

$$F = \Gamma^{-1}\eta, \tag{A.6}$$

$$\begin{aligned} \textit{where} \quad & F = \mathrm{vec}(F_1, \dots, F_n), \\ & \eta = \mathrm{vec}(S_{1\bullet} L \hat{\Theta}_1, \dots, S_{n\bullet} L \hat{\Theta}_n), \\ & \Gamma = [\Gamma_{ij}]_{i,j\in N}, \quad \textit{where } \Gamma_{ij} = \hat{\Sigma}_{ij} \otimes S_{ij}. \end{aligned}$$

Furthermore, the optimal performance is given by

$$J^* = \mathrm{Tr}(L^\mathsf{T} S L P_0) - \eta^\mathsf{T} \Gamma^{-1} \eta, \tag{A.7}$$

where $P_0 = \mathrm{var}(x - \hat{x}_0)$. □

The proof is presented in Section Appendix A.4.

Now, we state basic properties of Gaussian random variables.

Lemma 4 *Let a and b be jointly Gaussian zero-mean random variables with covariance $\left[\begin{smallmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{smallmatrix}\right]$. Then,*

1. $\mathbb{E}[a|b] = \Sigma_{ab} \Sigma_{bb}^{-1} b$
2. *For matrices A and B of appropriate dimensions,*

$$\mathbb{E}[a^T A^T B b] = \mathrm{Tr}(A \Sigma_{ab} B^\mathsf{T}) = \mathrm{Tr}(\Sigma_{ba} A^\mathsf{T} B). \quad \square$$

In order to compute the gains and the performance, we need to compute $\hat{\Theta}_i = \mathrm{cov}(x, \tilde{y}_i)$ and $\hat{\Sigma}_{ij} = \mathrm{cov}(\tilde{y}_i, \tilde{y}_j)$.

Lemma 5 *We have that*

1. $\hat{\Theta}_i = \Theta_i - \Theta_0 \Sigma_{00}^{-1} \Sigma_{0i}$.
2. $\hat{\Sigma}_{ij} = \Sigma_{ij} - \Sigma_{i0} \Sigma_{00}^{-1} \Sigma_{0j}$.

Proof We have

1. By definition, $\hat{y}_i = \Sigma_{i0} \Sigma_{00}^{-1} y_0$. Therefore,

$$\begin{aligned} \hat{\Theta}_i &= \mathrm{cov}(x, y_i - \Sigma_{i0} \Sigma_{00}^{-1} y_0) \\ &= \mathbb{E}[x (y_i - \Sigma_{i0} \Sigma_{00}^{-1} y_0)^\mathsf{T}] \overset{(a)}{=} \Theta_i - \Theta_0 \Sigma_{00}^{-1} \Sigma_{0i}. \end{aligned}$$

where (a) holds by the definition of Θ_i and Θ_0.

2. Using the equation for $\hat{y}_i$,

$$\hat{\Sigma}_{ij} = \mathbb{E}[(y_i - \hat{y}_i)(y_j - \hat{y}_j)^\mathsf{T}] = \Sigma_{ij} - \Sigma_{i0} \Sigma_{00}^{-1} \Sigma_{0j}. \ \blacksquare$$

Appendix A.3. Some preliminary results

Next we state some properties that are used in the proof of Theorem 4.

Lemma 6 *The following relations hold:*

1. $\hat{x}_0 = \Theta_0 \Sigma_{00}^{-1} y_0$.
2. $\hat{x}_i = \hat{x}_0 + \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i)$.
3. $\hat{x}_i - \hat{x}_0$ *is orthogonal to* y_0.
4. $\mathbb{E}[y_j - \hat{y}_j \,|\, y_0, y_i] = \hat{\Sigma}_{ji} \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i)$.
5. $\mathrm{cov}(x - \hat{x}_0, y_i - \hat{y}_i) = \hat{\Theta}_i$.

Proof We prove each part separately.

1. This follows from Lemma 4, part (1).
2. Note that we can write $\hat{x}_i$ as $\mathbb{E}[x \,|\, y_0, y_i - \hat{y}_i]$. Since y_0 and $y_i - \hat{y}_i$ are orthogonal, we have

$$\hat{x}_i = \mathbb{E}[x \,|\, y_0] + \mathbb{E}[x | y_i - \hat{y}_i] = \hat{x}_0 + \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i),$$

where the last equality follows from Lemma 4, part (1).

3. By part (2), $\hat{x}_i - \hat{x}_0$ is a linear function of $y_i - \hat{y}_i$, which is orthogonal to y_0 because of the orthogonal projection theorem.
4. Note that we can write $\mathbb{E}[y_j - \hat{y}_j \,|\, y_0, y_i]$ as $\mathbb{E}[y_j - \hat{y}_j \,|\, y_0, y_i - \hat{y}_i]$. Thus,

$$\begin{aligned} \mathbb{E}[y_j - \hat{y}_j | y_0, y_i] &= \mathbb{E}[y_j - \hat{y}_j | y_0] + \mathbb{E}[y_j - \hat{y}_j | y_i - \hat{y}_i] \\ &\overset{(a)}{=} \mathrm{cov}(y_j - \hat{y}_j, y_i - \hat{y}_i) \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i), \end{aligned}$$

where (a) uses part (3), Lemma 4, part (1), and the definition of $\mathrm{cov}(y_j - \hat{y}_j, y_i - \hat{y}_i)$.

5. We have the following

$$\mathrm{cov}(x - \hat{x}_0, y_i - \hat{y}_i) \overset{(a)}{=} \mathrm{cov}(x, y_i - \hat{y}_i) \overset{(b)}{=} \hat{\Theta}_i \quad \blacksquare$$

where (a) follows from the fact that $(y_i - \hat{y}_i)$ is orthogonal to y_0 and hence $\hat{x}_0$, and (b) follows from the definition of $\hat{\Theta}_i$.

Lemma 7 *For any $\{S_{ij}\}_{i\in N}$, $\{P_{ij}\}_{i\in N}$ and $\{L_i\}_{i\in N}$ of compatible dimensions, the following matrix equation*

$$\sum_{j\in N} \left[S_{ij} F_j P_{ji} - S_{ij} L_j P_{ii} \right] = 0, \quad \forall i \in N. \tag{A.8}$$

for unknown $\{F_i\}_{i\in N}$ of compatible dimensions can be written in vectorized form as

$$\Gamma F = \eta, \tag{A.9}$$

where

$$\begin{aligned} & F = \mathrm{vec}(F_1, \dots, F_n), \\ & \eta = \mathrm{vec}(S_{1\bullet} L P_{11}, \dots, S_{n\bullet} L P_{nn}), \\ & \Gamma = [\Gamma_{ij}]_{i,j\in N}, \quad \textit{where } \Gamma_{ij} = P_{ij} \otimes S_{ij}. \end{aligned}$$

Furthermore, if $S > 0$, $P \geq 0$, and $P_{ii} > 0$, $i \in N$, then $\Gamma > 0$ and thus invertible. Then, Eq. (A.8) *has a unique solution that is given by*

$$F = \Gamma^{-1}\eta. \tag{A.10}$$

□

Proof By vectorizing both sides of (A.8) and using $\mathrm{vec}(ABC) = (C^\intercal \otimes A) \times \mathrm{vec}(B)$, we get

$$\sum_{j\in N} (P_{ij} \otimes S_{ij})\,\mathrm{vec}(F_j) - \mathrm{vec}(S_{i\bullet} L P_{ii}) = 0, \quad \forall i \in N.$$

Substituting $\Gamma_{ij} = P_{ij} \otimes S_{ij}$ and $\eta_i = \mathrm{vec}(S_{i\bullet} L P_{ii})$, we get (A.9).

If $S > 0$, $P \geq 0$, and $P_{ii} > 0$, $i \in N$, then [10, Lemma 1] implies that $\Gamma > 0$ and thus invertible. Hence, Eq. (A.8) has a unique solution that is given by (A.10). ■

Appendix A.4. Proof of Theorem 4

Given agent $i \in N$, let (g_i, g_{-i}) and $(\hat{z}_i, \hat{z}_{-i})$ denote the strategy and estimates of all agents. Then according to [40, Theorem 3], a necessary and sufficient condition for a strategy (g_i, g_{-i}) to be team optimal is

$$\frac{\partial}{\partial \hat{z}_i} \mathbb{E}^{g_{-i}}[c(x, \hat{z}_i, \hat{z}_{-i}) | y_0, y_i] = 0, \quad \forall i \in N. \tag{A.11}$$

From dominated convergence theorem, we can interchange the order of derivative and expectation to get

$$\begin{aligned}
\text{LHS of (A.11)} &= \mathbb{E}^{g_{-i}}\left[\frac{\partial}{\partial \hat{z}_i} c(x, \hat{z}_i, \hat{z}_{-i}) \,\middle|\, y_0, y_i\right] \\
&= \mathbb{E}^{g_{-i}}\left[\frac{\partial}{\partial \hat{z}_i} \sum_{k\in N}\sum_{j\in N} (L_k x - \hat{z}_k)^\intercal S_{kj} (L_j x - \hat{z}_j) \,\middle|\, y_0, y_i\right] \\
&= 2\mathbb{E}^{g_{-i}}\left[\sum_{j\in N} S_{ij}(L_j x - \hat{z}_j) \,\middle|\, y_0, y_i\right]
\end{aligned}$$

Substituting the above in (A.11), we get that a necessary and sufficient condition for a strategy (g_i, g_{-i}) to be team optimal is

$$\sum_{j\in N} \Big[S_{ij} \mathbb{E}[\hat{z}_j \,|\, y_0, y_i] - S_{ij} L_j \mathbb{E}[x \,|\, y_0, y_i] \Big] = 0, \quad \forall i \in N. \tag{A.12}$$

Thus, the strategy g given by (A.4) is optimal if and only if

$$\sum_{j\in N} \Big[S_{ij} \mathbb{E}\big[F_j (y_j - \hat{y}_j) + L_j \hat{x}_0 \;\big|\; y_0, y_i\big] - S_{ij} L_j \mathbb{E}\big[x \;\big|\; y_0, y_i\big] \Big] = 0, \quad \forall i \in N, \tag{A.13}$$

or equivalently

$$\sum_{j\in N} \Big[S_{ij} F_j \mathbb{E}\big[(y_j - \hat{y}_j) | y_0, y_i\big] - S_{ij} L_j \mathbb{E}\big[x - \hat{x}_0 \;\big|\; y_0, y_i\big] \Big] = 0. \quad \forall i \in N. \tag{A.14}$$

Note that from Lemma 6 (2),

$$\mathbb{E}[x | y_0, y_i] - \mathbb{E}[\hat{x}_0 | y_0, y_i] = \hat{x}_i - \hat{x}_0 = \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i).$$

Substituting the above and the expression for $\mathbb{E}[y_j - \hat{y}_j | y_0, y_i]$ from Lemma 6 in (A.14), we get that for all $i \in N$,

$$\sum_{j\in N} \Big[S_{ij} F_j \hat{\Sigma}_{ji} \hat{\Sigma}_{ii}^{-1} - S_{ij} L_j \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \Big] (y_i - \hat{y}_i) = 0.$$

Since the above should hold for all $y_i - \hat{y}_i \in \mathbb{R}^{d_y^i}$, the coefficient of $(y_i - \hat{y}_i)$ must be identically zero. Thus,

$$\sum_{j\in N} \Big[S_{ij} F_j \hat{\Sigma}_{ji} \hat{\Sigma}_{ii}^{-1} - S_{ij} L_j \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \Big] = 0, \quad \forall i \in N. \tag{A.15}$$

Lemma 8 *If $\hat{\Sigma}_{ii} > 0$, then* (A.15) *has a unique solution.*□

Proof The result follows from Lemma 7.

Now for the optimal value of the estimation cost, consider a single term of the cost

$$\begin{aligned}
&\mathbb{E}[(L_i x - \hat{z}_i)^\intercal S_{ij} (L_j x - \hat{z}_j)] \\
&\overset{(a)}{=} \mathbb{E}\big[(x - \hat{x}_0)^\intercal L_i^\intercal S_{ij} L_j (x - \hat{x}_0) \\
&\qquad - 2(y_i - \hat{y}_i)^\intercal F_i^\intercal S_{ij} L_j (x - \hat{x}_0) \\
&\qquad + (y_i - \hat{y}_i)^\intercal F_i^\intercal S_{ij} F_j (y_j - \hat{y}_j)\big] \\
&\overset{(b)}{=} \mathrm{Tr}(P_0 L_i^\intercal S_{ij} L_j) - 2\,\mathrm{Tr}(\hat{\Theta}_i F_i^\intercal S_{ij} L_j) + \mathrm{Tr}(\hat{\Sigma}_{ij}^\intercal F_i^\intercal S_{ij} F_j) \\
&\overset{(c)}{=} \mathrm{Tr}(P_0 L_i^\intercal S_{ij} L_j) - 2\,\mathrm{Tr}(F_i^\intercal S_{ij} L_j \hat{\Theta}_i) + \mathrm{Tr}(F_i^\intercal S_{ij} F_j \hat{\Sigma}_{ji}),
\end{aligned} \tag{A.16}$$

where (a) follows from substituting (A.4), (b) uses Lemma 4, part (2) and Lemma 6, part (5), and (c) uses the fact that for any matrices $\mathrm{Tr}(ABCD) = \mathrm{Tr}(BCDA)$. Thus, the total cost is

$$\begin{aligned}
J^* &= \sum_{i\in N}\sum_{j\in N} \mathbb{E}[(L_i x - \hat{z}_i)^\intercal S_{ij} (L_j x - \hat{z}_j)] \\
&\overset{(d)}{=} \sum_{i\in N}\sum_{j\in N} \Big[\mathrm{Tr}(P_0 L_i^\intercal S_{ij} L_j) - 2\,\mathrm{Tr}(F_i^\intercal S_{ij} L_j \hat{\Theta}_i) \\
&\qquad\qquad + \mathrm{Tr}(F_i^\intercal S_{ij} F_j \hat{\Sigma}_{ji}) \Big] \\
&= \mathrm{Tr}(P_0 L^\intercal S L) \\
&\quad - \sum_{i\in N} \mathrm{Tr}\left(F_i^\intercal \sum_{j\in N} \Big[2 S_{ij} L_j \hat{\Theta}_i - S_{ij} F_j \hat{\Sigma}_{ji} \Big] \right) \\
&\overset{(e)}{=} \mathrm{Tr}(P_0 L^\intercal S L) - \sum_{i\in N} \mathrm{Tr}\left(F_i^\intercal \sum_{j\in N} S_{ij} L_j \hat{\Theta}_i \right)
\end{aligned} \tag{A.17}$$

where (d) follows from (A.16), and (e) follows from (A.15). The result now follows from observing that

$$\begin{aligned}
\sum_{i\in N} \mathrm{Tr}\left(F_i^\intercal \sum_{j\in N} S_{ij} L_j \hat{\Theta}_i \right) &= \sum_{i\in N} \mathrm{Tr}(F_i^\intercal S_i L \hat{\Theta}_i) \\
&= \sum_{i\in N} \mathrm{vec}(F_i)^\intercal \mathrm{vec}(S_i L \hat{\Theta}_i) = F^\intercal \eta = \eta^\intercal \Gamma^{-1} \eta,
\end{aligned}$$

where the first equality follows from $\mathrm{Tr}(A^\intercal B) = \mathrm{vec}(A)^\intercal \mathrm{vec}(B)$.

Appendix B. One-step delayed observation sharing

Appendix B.1. Problem statement

In this section, we use the result of Theorem 1 to show the relationship between team optimal decentralized estimation and control in delayed observation sharing model [10, 11, 46]. The notation used in this section is self-contained and consistent with the standard notation used in decentralized stochastic control.

Consider a decentralized control system with n agents, indexed by the set $N = \{1, \dots, n\}$. The system has a state $x(t) \in \mathbb{R}^{d_x}$. The initial state $x(1) \sim N(0, \Sigma_x)$ and the state evolves as follows:

$$x(t+1) = A(t)x(t) + B(t)u(t) + w(t), \tag{B.1}$$

where A and B are matrices of appropriate dimensions. $u(t) = \mathrm{vec}(u_1(t), \cdots, u_n(t))$, where $u_i(t) \in \mathbb{R}^{d_u^i}$ is the control action chosen by agent i, and $\{w(t)\}_{t\ge1}$, $w(t) \in \mathbb{R}^{d_x}$ is an i.i.d. process with $w(t) \sim \mathcal{N}(0, \Sigma_w)$. Each agent observes a noisy version $y_i(t) \in \mathbb{R}^{d_y^i}$ of the state given by

$$y_i(t) = C_i(t)x(t) + v_i(t) \tag{B.2}$$

where $\{v_i(t)\}_{t\ge1}$, $v_i(t) \in \mathbb{R}^{d_y^i}$, is an i.i.d. process with $v_i(t) \sim (0, \Sigma_v^i)$. This may be written in a vector form as

$$y(t) = C(t)x(t) + v(t), \tag{B.3}$$

where $C = \mathrm{rows}(C_1, \dots, C_n)$, $v(t) = \mathrm{vec}(v_1(t), \dots, v_n(t))$, and $y(t) = \mathrm{vec}(y_1(t), \dots, y_n(t))$.

Assumption 1: The primitive random variables ($x(1)$, $\{w(t)\}_{t\ge1}, \{v_1(t)\}_{t\ge1}, \dots, \{v_n(t)\}_{t\ge1}$) are independent.

In addition to its local observation $y_i(t)$, each agent also receives the one-step delayed observations of all agents. Thus, the information available to agent i is given by

$$I_i(t) \coloneqq \{y_i(t), y(1{:}t-1), u(1{:}t-1)\}. \tag{B.4}$$

Therefore, agent i chooses the control action $u_i(t)$ as follows.

$$u_i(t) = g_{i,t}(I_i(t)), \tag{B.5}$$

where $g_{i,t}$ is the control laws of agent i at time t. The collection $g = (g_1, \dots, g_n)$, where $g_i = (g_{i,1}, \dots, g_{i,T})$ is called the control strategy of the system. The performance of any control strategy g is given by

$$J(g) = \mathbb{E}^g\Big[\sum_{t=1}^{T-1}\big[x(t)^\intercal Q x(t) + u(t)^\intercal R u(t)\big] + x(T)^\intercal Q x(T)\Big], \tag{B.6}$$

where Q is symmetric positive semi-definite matrix, R is symmetric positive definite matrix, and the expectation is with respect to the joint measure on the system variables induced by the choice of g.

Problem 4 Given the system dynamics and the noise statistics, choose a control strategy g to minimize the total cost $J(g)$ given by (B.6).

Problem 4 is a decentralized stochastic control problem. In such problems there is no separation of estimation and control (see, for example [10]). We show that this lack of separation is due to the fact that the team optimal decentralized estimation strategy depends on the weight matrix of the estimation cost.

Appendix B.2. Equivalence to team optimal decentralized estimation

We start with a basic property of linear quadratic models. Let $P(1{:}T)$ denote the solution to the following backward Riccati equation. $P(T) = Q$ and for $t \in \{T-1, \dots, 1\}$,

$$P(t) = Q + A^\intercal P(t+1)A - A^\intercal P(t+1)B(R + B^\intercal P(t+1)B)^{-1}B^\intercal P(t+1)A.$$

Define

$$S(t) = R + B^\intercal P(t+1)B,$$
$$L(t) = S(t)^{-1}(B^\intercal P(t+1)A).$$

Then, we have the following.

Lemma 9 *For any control strategy g, define*

$$J^\circ(g) = \sum_{t=1}^{T-1}\mathbb{E}[(u(t) + L(t)x(t))^\intercal S(t)(u(t) + L(t)x(t))]. \tag{B.7}$$

Then, a strategy g that minimizes $J^\circ(g)$ also minimizes $J(g)$. □

PROOF Following [47, Chapter 8, Lemma 6.1], we can show that the total cost $J(g)$ can be written as

$$J(g) = \sum_{t=1}^{T-1}\mathbb{E}\big[w(t)^\intercal P(t+1)w(t) + x(1)^\intercal P(1)x(1)\big] + \sum_{t=1}^{T-1}\mathbb{E}\big[(u(t) + L(t)x(t))^\intercal S(t)(u(t) + L(t)x(t))\big]. \tag{B.8}$$

The third term is equal to $J^\circ(g)$ and the first two terms do not depend on the control strategy g. Thus, $J(g)$ and $J^\circ(g)$ have the same argmin. ■

Now, we split the state $x(t)$ into a deterministic part $\bar{x}(t)$ and a stochastic part $\tilde{x}(t)$ as follows. $\bar{x}(1) = 0$, $\tilde{x}(1) = x(1)$, and

$$\bar{x}(t+1) = A\bar{x}(t) + Bu(t), \quad \tilde{x}(t+1) = A\tilde{x}(t) + w(t),$$
$$\bar{y}(t) = C\bar{x}(t), \qquad \tilde{y}(t) = C\tilde{x}(t) + v(t).$$

Since the system is linear, we have

$$x(t) = \bar{x}(t) + \tilde{x}(t) \quad \text{and} \quad y(t) = \bar{y}(t) + \tilde{y}(t).$$

Note that $\bar{x}(t)$ is a function of the past control actions, which are known to all agents. Now, for any control strategy g, define $\hat{z}_i(t) = u_i(t) + L_i(t)\bar{x}(t)$. Then, the cost $J^\circ(g)$ may be written as

$$\sum_{t=1}^{T-1} \mathbb{E}[(\hat{z}_i(t) + L(t)\tilde{x}(t))^\intercal S(t)(\hat{z}_i(t) + L(t)\tilde{x}(t))]. \quad \text{(B.9)}$$

The process $\{\tilde{x}(t)\}_{t\geq 1}$ is an uncontrolled linear stochastic process and the cost (B.9) is of of the same form as the weighted mean-square cost that we have considered in this paper.

Following [3], we define $\tilde{I}_i(t) = \{\tilde{y}_i(t), \tilde{y}(1{:}t-1)\}$ which may be considered as the control-free part of the information structure.

Lemma 10 *For any strategy g and any agent $i \in N$, $\tilde{I}_i(t)$ is equivalent to $I_i(t)$, i.e., they generate the same sigma algebra.* □

Proof The result follows from a similar argument as given in [48, Chapter 7, Section 3]. ■

Since $\tilde{I}_i(t)$ is equivalent to $I_i(t)$, we may assume that $\hat{z}_i(t)$ is chosen as a function of $\tilde{I}_i(t)$ instead of $I_i(t)$. Thus, Problem 4 is equivalent to the following team optimal decentralized state estimation problem.

Problem 5 Suppose n agents observe the linear dynamical system $\{\tilde{x}(t)\}_{t\geq 1}$ and share their observations over a one-step delayed sharing communication graph. Thus, the information available at agent i is

$$\tilde{I}_i(t) = \{\tilde{y}_i(t), \tilde{y}(1{:}t-1)\}.$$

Agent i chooses an estimate $\hat{z}_i(t)$ of $\tilde{x}(t)$ according to an estimation strategy $h_{i,t}$, i.e.,

$$\hat{z}_i(t) = h_{i,t}(\tilde{I}_i(t))$$

to minimize an estimation cost given by (B.9).

Problem 5 is a decentralized state estimation problem and can be solved using Theorem 1 and 2. One can then take the solution of Problem 5 and translate it back to Problem 4 as follows.

Theorem 5 *Let h^* be the optimal strategy for Problem 5, i.e.,*

$$\begin{aligned} h^*_{i,t}(\tilde{I}_i(t)) &= -L_i(t)\hat{\tilde{x}}(t) \\ &\quad - F_i(t)\Big(\tilde{y}_i(t) - \mathbb{E}[\tilde{y}_i(t)|\tilde{y}(1{:}t-1)]\Big), \end{aligned} \quad \text{(B.10)}$$

where

$$\begin{aligned} \hat{\tilde{x}}(t) &= \mathbb{E}[\tilde{x}(t)|\tilde{y}(1{:}t-1)], \\ L(t) &= \operatorname{rows}(L_1(t), \dots, L_n(t)), \end{aligned}$$

and the gains $\{F_i(t)\}$ are computed as per Theorem 1. Define strategy g^* as follows:

$$g^*_{i,t}(I_i(t)) = h^*_{i,t}(\tilde{I}_i(t)) - L_i(t)\bar{x}(t), \quad \text{(B.11)}$$

i.e.,

$$\begin{aligned} &g^*_{i,t}(I_i(t)) = -L_i(t)\hat{x}(t) \\ &\quad - F_i(t)\Big(y_i(t) - \mathbb{E}[y_i(t)|y(1{:}t-1), u(1{:}t-1)]\Big), \end{aligned} \quad \text{(B.12)}$$

where $\hat{x}(t) = \mathbb{E}[x(t)|I^{\text{com}}(t)] = \bar{x}(t) + \mathbb{E}[\tilde{x}(t)|\tilde{y}(1{:}t-1)]$. Then g^* is the optimal strategy for Problem 4. □

Proof The change of variables $\hat{z}_i(t) = u_i(t) + L_i(t)\bar{x}(t)$ implies that if h^* is an optimal strategy for Problem 5, then g^* given by (B.11) is optimal for Problem 4.

To establish (B.12), we need to show that $\hat{x}(t) = \bar{x}(t) + \hat{\tilde{x}}(t)$. Define, $I^{\text{com}}(t) = \{y(1{:}t-1), u(1{:}t-1)\}$ and $\tilde{I}^{\text{com}}(t) = \{\tilde{y}(1{:}t-1)\}$. Then by Lemma 10 we have, $I^{\text{com}}(t)$ is equivalent to $\tilde{I}^{\text{com}}(t)$, i.e., they generate the same sigma algebra. The rest of the proof follows from the definition of $\hat{x}(t)$. We have

$$\begin{aligned} \hat{x}(t) &= \mathbb{E}[x(t)|\tilde{I}^{\text{com}}(t)] \\ &\overset{(a)}{=} \mathbb{E}[\bar{x}(t)|I^{\text{com}}(t)] + \mathbb{E}[\tilde{x}(t)|\tilde{I}^{\text{com}}(t)] \\ &\overset{(b)}{=} \bar{x}(t) + \hat{\tilde{x}}(t), \end{aligned}$$

where (a) follows from state splitting and $I^{\text{com}}(t) = \tilde{I}^{\text{com}}(t)$ and (b) follows from the fact that $\bar{x}(t)$ is a deterministic function of $I^{\text{com}}(t)$. ■

The main take away is as follows. By a simple change of variables we showed that the one-step delayed observation sharing problem is equivalent to a decentralized state estimation problem, where the weight matrix $S(t)$ of the estimation cost depends on the backward Riccati equation for the cost function. The team optimal decentralized estimation strategy depends on the weight matrix $S(t)$ and that is the reason why there is no separation between estimation and control. Nonetheless, the optimal gains can be computed as follows.

1. Solve a Riccati equation to compute the weight functions $S(1{:}T)$ and gains $L(1{:}T)$.
2. Solve a Kalman filtering equation (which does not depend on $S(1{:}T)$) to compute the covariances $\hat{\Sigma}(t)$ and $\hat{\Theta}(t)$ defined in Theorem 1.
3. Use $S(t)$, $L(t)$, $\hat{\Sigma}(t)$, and $\hat{\Theta}(t)$ to obtain the optimal gains $F_i(t)$ by solving a system of matrix equations.
4. Using Theorem 5 above, we can write the optimal strategy $g^*_{i,t}$ in terms of $F_i(t)$ and $L_i(t)$.