Optical Flow

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1 Introduction

The Optical Flow comprizes the problem of *estimation the motion field from image sequences*, using the information extracted from the spatial and temporal variations of the image brightness. Within some limitations optical flow field is an accurate estimate of motion field. Hence, optical flow can be referred as *the distribution of apparent velocities of movement of brightness patterns in an image* (Horn and Schunk, 1981). The approximation to the 2-D motion field, a projection of 3D velocities from the moving scene points onto the image plane, using spatiotemporal patterns of image intensity is being computed.

Generally, wide variety of tasks could be done exploiting properties and point values of optical flow field, like image segmentation, etc.

In order to connect brightness variations and motion field, we introduce the assumption regarding brightness constancy- and, hence, we derive an *image brightness constancy equation*. But, to be able to do that, we have to make one more assumption, and that is *the image brightness is continous and differentiable as many times as needed in both spatial and temporal domain.*

2 The Image Brightness Constancy Equation

The apparent brightness of moving objects, generally remains constant. Since, the image irradiance is proportional to the scene radiance in the direction of the optical axis of the camera, then for assumed constant proportionality factor across the entire image plane, apparent brightness constancy can be expressed through the image brightness constancy eqaution, regarding the temporal dimension:

$$\frac{d}{dt}E(x(t),(y(t)) = 0 \Rightarrow \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial t} = 0 \tag{1}$$

$$\nabla E = [\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}]^T \tag{2}$$

$$\vec{v} = [v_x, v_y] = [\frac{dx}{dt}, \frac{dy}{dt}] \tag{3}$$

The Image Brightness Constancy Equation:

$$(\nabla E)^T \vec{v} + \frac{\partial E}{\partial t} = 0 \tag{4}$$

3 The Aperture Problem

The last equation in the last section helps to understand the problem we are going to introduce now. Since we have dot product of image brightness spatial gradient and apparent velocity, it is like we are projecting the apparent velocity onto the direction

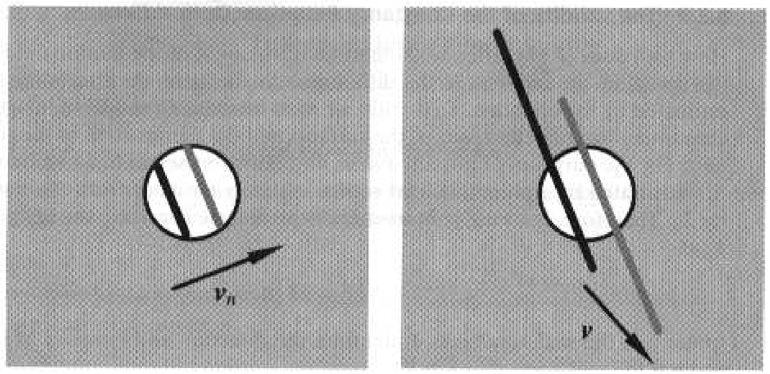

Figure 1: The aperture problem: the black and gray lines show two positions of the same image line in two consecutive frames. The image velocity percieved in the first frame (on the left) through the small aperture $\vec{v}$, is only the component parallel to the image gradient of the true image velocity $\vec{v}$, revealed in the second frame (on the right)

of the gradient in question, and, thus, it is easy to see that we are able to determine only the velocity component in that direction.

$$\frac{\frac{\partial E}{\partial t}}{\|\nabla E\|} = -\frac{\nabla E^T vecv}{|\nabla E\|} = -\vec{v}_n \tag{5}$$

This is not a surprising result, because, after all, we have only one equation (*brightness constancy equation*), and need to determine two velocity components. This leads us to the conclusion that we need one more constraint.

4 Lambertian Model and Validity of the Constancy Equation

In order to estimate the accuracy of the derived equation for calculating the normal component of the motion field, we will find the diference, $\nabla\vec{v}$ between tre true value and one that's been estimated.

We are assuming that the illuminated surface is Lambertian, and hence, we have:

$$E = \rho\vec{I}^T\vec{n} \tag{6}$$

So how true is the assumption $\frac{\partial E}{\partial t} = (\nabla\vec{E}^T$, given the Lambertian conditions.

If we apply temporal derivative to both sides of the image irradiance equation, we get:

$$\frac{dE}{dt} = \rho\vec{I}^T\frac{d\vec{n}}{dt} \tag{7}$$

If the surface is moving relative to the camera with translational velocity $\vec{T}$, and angular velocity $\vec{\omega}$, then, taking into account that$\vec{T}$ has no influence on the$\frac{d\vec{n}}{dt}$, we can find the relation between derivative of surface normal with respect to time, as a function of only angular velocity. But, let us first examine wath follows.

For some point $\vec{P}$ on the surface that is moving with translational velocity $\vec{T}$, and angular velocity $\vec{\omega}$ we can determine the overall velocity as:

$$\vec{v} = -\vec{T} - \vec{\omega} \times \vec{P} \tag{8}$$

If we substitute $\vec{P}$ with $\vec{n}$, and observe the change of normal vector, in two close time points, separated by ∇t interval, namely- in $t = 0$ we have$\vec{n}_0$, and in $t = t_0 + \nabla t$, we have $\vec{n}_{\nabla t} = \vec{n}_0 + (-\vec{\omega} \times \vec{n}_0)$

$$\frac{\vec{n}_{\nabla t} - \vec{n}_0}{\nabla t} = -\frac{\vec{\omega} \times \vec{n}_0}{\nabla t} \Rightarrow \frac{\vec{n}_0 - \vec{n}_{\nabla t}}{\nabla t} = \frac{\vec{\omega} \times \vec{n}_0}{\nabla t} \tag{9}$$

If we let $\nabla t \to 0$, then we have:

$$\frac{d\vec{n}_0}{dt} = \vec{\omega} \times \vec{n}_0 \Rightarrow \frac{dE}{dt} = \rho \vec{I}^T (\vec{\omega} \times \vec{n}) \tag{10}$$

Now, using brightness constancy model, we can estimate the error. The error will be reflected through $\frac{\partial E}{\partial t}$

We have that the total temporal derivative of E is:

$$\frac{dE}{dt} = \rho \vec{I}^T (\vec{\omega} \times \vec{n}) = (\nabla E)^T \vec{v} + \frac{\partial E}{\partial t} \tag{11}$$

Our estimated normal velocity is given by $\vec{v}_n = \frac{\rho \vec{I}^T (\vec{\omega} \times \vec{n})}{\|\nabla E\|}$, and error is given by the term $\frac{-\partial E}{\partial t}$ normalized with the magnitude of vector ∇E:

$$\mid \nabla \vec{v} \mid = \frac{\frac{-\partial E}{\partial t}}{\|\nabla E\|} = \frac{\rho \vec{I}^T (\vec{\omega} \times \vec{n})}{\|\nabla E\|} \tag{12}$$

Discussing the validity of *brightness constancy assumption*- we find that it is a good assumption for the error which converges to zero that happens for: $\vec{\omega} = 0$, or if $(\vec{\omega} \times \vec{n})$ is orthogonal to the $\vec{I}$, or if $\vec{\omega}$ is parallel to $\vec{n}$).

4.1 Observations

- In general, the constant brightness assumption under the Lambertian model, reveals only the true normal velocity $\vec{v}_n$ when $\vec{\omega} = 0$ (pure translation)

- The error becomes small for great values of$\|\nabla E\|$, and that leads to the edge points.

we have the whole set of successive image frames (so that we have $E, \frac{\partial E}{\partial t}$), and we are after $\vec{v}$...we are actually after $\vec{v}_n$, good enough to satisfy the basic equation $\frac{\partial E}{\partial t} + (\nabla E)\vec{v} = 0$

5 Estimation of the Motion Field- Techniques

Techniques used in estimation of the motion field, though numerous, can be roughly divided into to main categories:*differential techniques* and *matching techniques*.

5.0.1 Differential Approach

- use derivatives
- we have dense estimation field
- can be numerically sensitive

5.0.2 Feature Based (Matching)Approaches

- tracking features
- some techniques use Kalman filters (stochastic estimation and prediction techniques)
- sparce optical flow field
- more stable computationally, because they don't have to deal with problems introduce by dense fields

6 Optical Flow Algorithm

Regarding differential approaches, we could rather, instead of looking for a dense solution, relax that aim a little bit and build in the smoothness constraint. Here we notice a single dominant motion over a small neighborhood, so that we can assume two things:

- Constant Brightness (this is a strong assumption for a sequences of more than a few images)
- Over a window, or a patch (typical size of a patch is 5x5), there is a single $\vec{v}$ for all the points inside the patch

Now we can setup a functional:

$$\psi(\vec{v}) = \sum_{p_i \in Q} (\frac{\partial E}{\partial t} + (\nabla E^T \vec{v}))^2 \tag{13}$$

Q is a NxN patch, $\vec{p_i}, i = 1, 2, \ldots, NxN$ is the set of points inside the Q (in general, Q doesn't have to be rectangular, or square). We are calculating the spatial and temporal derivatives over all points p_i in the patch, assuming that the velocity vector is constant for all points in the particular patch.

$\psi(\vec{v})$ gives some kind of confidence measure, it behaves like a energy functional. This is the standard least square problem:

Find $\vec{v}$ that minimizes $\psi(\vec{v})$

Minimization is done in parallel- the algorithm could be improved a lot if we introduce a smoothness constraint over the optical flow field.

6.1 The Solution for The Lest Square Problem

We are solving for $\vec{v}$ over a single patch, for other Q_j, over the entire image plane, it would be done similarly.

The solution for this least square problem is given by the following system:

$$\underbrace{A^T A}_{(2\times2)} \underbrace{\vec{v}}_{(2\times1)} = \underbrace{A^T}_{(2\times N^2)} \underbrace{\vec{b}}_{(N^2\times1)} \Rightarrow \vec{v} = (A^T A)^{-1} A^T \vec{b} \tag{14}$$

$$A = \begin{bmatrix} \nabla E(\vec{p}_1) \\ \nabla E(\vec{p}_2) \\ \vdots \\ \nabla E(\vec{p}_{N^2}) \end{bmatrix}_{N^2\times2} \qquad \vec{b} = -\begin{bmatrix} \frac{\partial E}{\partial t}(\vec{p}_1) \\ \frac{\partial E}{\partial t}(\vec{p}_2) \\ \vdots \\ \frac{\partial E}{\partial t}(\vec{p}_{N^2}) \end{bmatrix}_{N^2\times1} \tag{15}$$

(14): Thie equation is given assuming that E is smooth enough so that we can compute derivatives,but, $(A^T A)^{-1}$ could be problem if $A^T A$ is rank defficient.

The structure of $(A^T A)$ is important- should not be singular!

Note: there is a problem of *occlusion* (no iso-contours in successive images that correspond to each other) as well- it might give a rise to the situation where we could loose a matching region for a patch in successive images, what means that our region of interest could be occluded in one of the frames, and then we loose the information about the region in particular time point. But, it is also of interest to find boundaries where we come across the changes, because in that way we could detect object boundaries and thus, achieve image segmentation.