Fourier Signature

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1 Beat

Recall the stuff about RMS voltage we did at the beginning of the semester. Now consider the sum of two sine waves of constant amplitude a and b, respectively, and slightly different (angular) frequency $\omega_1 > \omega_2$. One wave will periodically overtake the other. Imagine the wave amplitudes are represented by the sum of the x components of two vectors, length a and b, rotating, with their "tails" on the origin, at constant angular velocities ω_1 and ω_2, respectively. The resultant vector magnitude will vary between $a \pm b$. Examine Fig. 1. Since the *relative* angular velocity of the two constant frequency waves is $\omega_1 - \omega_2$, the time for one wave to "lap", *i.e.*, gain 2π on, the slower is $2\pi/(\omega_1 - \omega_2)$. The result is not harmonic but if $\omega_1 - \omega_2$ is very small compared to $\omega_1 + \omega_2$ it is *nearly* harmonic. If $a = b$ the time varying amplitude x is

$$x - a\sin\omega_1 t + a\sin\omega_2 t = 2a\left[\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t\right]\sin\left(\frac{\omega_1 + \omega_2}{2}\right)t$$

When ω_1 is only slightly greater than ω_2 the sine term will go through many cycles during the time it takes the cosine term to complete one. We can thus think of the wave (motion) as a sine wave

$$\sin\frac{(\omega_1 + \omega_2)t}{2}$$

with a slowly varying amplitude of

$$2a\cos\frac{(\omega_1 - \omega_2)t}{2}$$

The rhythmic variation in amplitude between 0 and $2a$ is called *beating*. The phenomenon, like train whistles and Doppler effect, could be more easily experienced years ago when twin *piston* engine aircraft flying nearby were common and the engine sound produced by two motors running at just *slightly* different revs would vary in an irregular cyclic manner as the difference $\omega_1 - \omega_2$ between the two engine speeds varied. You can still hear this in old war movies. ("Movies" is an archaic word; short for "moving pictures". Today we say "film" but that too will soon become meaningless as the digital revolution proceeds. In a way, this is "aliasing", a subject that was introduced in the oscilloscope lab and elsewhere in Professor Buehler's notes. A better example of aliasing is the illusion of backward-turning stage-coach wheels in cowboy films. A *practical* application of this phenomenon is angular velocity measurement by means of the stroboscope. This too would make a fine question on the final exam.)

2 Fourier Series

Consider all possible sine and cosine wave combinations of integer multiples of some fundamental, "slowest" angular displacement $x = \omega t$.

$$\begin{aligned} y(x) = & \quad a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \\ & +b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \end{aligned} \tag{1}$$

where $x = \omega t$. The sines are odd functions because they are antisymmetric about $x = 0$, *i.e.*, the y-axis, so that $y(x) = -y(-x)$. You can easily see why the cosine terms are the even ones. b_0 is the zeroth order term, zero is the first term in the infinite series expansion that produces cosine and zero is even. Cosine is therefore symmetric. Multiply both sides of Eq. 1 by $\sin nx$ and integrate. ...I know, I know, I told you not to integrate through discontinuities but this implies piecewise.

$$\int_0^{2\pi} y \sin nx dx = a_n \pi \tag{2}$$

Similarly, multiplying by $\cos nx$ and integrating produces the cosine coefficients.

$$\int_0^{2\pi} y \cos nx dx = b_n \pi \tag{3}$$

Note that this works when $n = 0$ to give the DC offset.

$$\int_0^{2\pi} y dx = 2\pi b_0 \tag{4}$$

So if the function $y = y(x)$ is known, given, measured, *etc.*, one may characterize it in terms of its Fourier coefficients to whatever precision is deemed practical. To get a feel for this, note that

- b_0 = average of *absolute* value of y for the period $x = 2\pi$,
- b_n = average of *absolute* value of $y \cos nx$ for this period and
- a_n = average of *absolute* value of $y \sin nx$ for this period.

3 Practical (Statistical?) Application

Often $y(x)$ is unknown but we have, say, a stethoscopic recording of sounds, vibrations, *etc.*. In these cases one may conveniently and efficiently find individual a_n and b_n by numerical methods. To determine, say, a_n, look at Eq. 1. The curve y is multiplied by $\sin nx$ which is repeated every $2\pi/n$. Divide the curve into n parts and add the n sections together before multiplying. The coefficients are then determined analytically as

$$a_n = \frac{1}{\pi} \int_0^{2\pi/n} Y \sin nx dx$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi/n} Y \cos nx dx$$

where Y is the result of the addition of the n segments. For numerical analysis, the interval $2\pi/n$ is divided into h equal parts and we determine the values of Y and sin or $\cos nx$ for these subintervals. The products, say, $Y \sin nx$ are then summed up and multiplied by the subinterval width $dx = 2\pi/hn$. The numerical procedure is

$$a_n = \frac{2}{hn} \sum_{k=1}^{h} Y_k \sin k \left(\frac{2\pi}{h} \right) \tag{5}$$

$$b_n = \frac{2}{hn} \sum_{k=1}^{h} Y_k \cos k \left(\frac{2\pi}{h} \right) \tag{6}$$

The more irregular the curve, the more intervals h must be taken. Much of modern signal processing's success is due to efficient ways to do this, *e.g.*, with various "flavours" of FFT=Fast Fourier Transform. One can also do cool morphing of images with Fourier encoding. After all anything finite may be assumed to represent a periodic cycle. Don't worry, consider the following simple *analytical* exercise and then do the next *numerical* one, with Maple and colleagues if desired, and all will be clear and obvious.

4 Example

Determine the Fourier series of the periodic triangular wave of maximum amplitude +1.0 and minimum amplitude 0. A typical period may be visulaized as an isosceles triangle whose base spans $-\pi \to +\pi$ on the x-axis with apex at point $(0, 1)$ as shown in the middle diagram on Fig. 1.

- Since this is symmetric about the origin, there are no sine terms.

$$y = b_0 + b_1 \cos x + b_2 \cos 2x + \cdots$$

- Since the waveform is simple we can use the analytic forms, represented by Eqs. 3 and 4, to determine the coefficients of the Fourier expansion.

- These are

$$y = \frac{1}{\pi}(x + \pi), \quad -\pi < x < 0$$
$$y = \frac{1}{\pi}(\pi - x), \quad 0 < x < \pi$$

- The first coefficient, b_0, is

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{0} \frac{1}{\pi}(x + \pi) dx + \frac{1}{2\pi} \int_{0}^{\pi} \frac{1}{\pi}(\pi - x) dx = \frac{1}{2}$$

  which agrees nicely with the average amplitude, *i.e.*, DC offset, of the wave.

- For $n \neq 0$ symmetry still applies and the two integrals in the sum combine.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{\pi}(\pi - x)\cos nx dx \\ &= \frac{2}{\pi}\int_0^\pi \cos nx dx - \frac{2}{\pi^2}\int_0^\pi x\cos nx dx \\ &= \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^\pi - \frac{2}{\pi^2}\left[\frac{\cos nx}{n^2} + \frac{x\sin nx}{n}\right]_0^\pi \\ &= 0 \text{ when } n \text{ is even and } \frac{4}{n^2\pi^2} \text{ when } n \text{ is odd} \end{aligned}$$

- Finally we can write the Fourier series.

$$y = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

5 Your Problem

Your task is to compute, plot and compare the approximating function, using the harmonic coefficients up to the fourth, *i.e.*, $a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4$, of the periodic wave shown in the lower diagram on Fig. 1. The ordinates of the curve in $10\,^\circ \equiv 36$ increments have been obtained experimentally, *e.g.*, using a piezo-accelerometer transducer to obtain a vibration (stethoscopic) recording from some piece of machinery which seems to be running with excessive vibration due to suspected imbalance of some rotating components, and are tabulated as 36 corresponding values. Demonstrated below is the procedure to get the third sine coefficient a_3. The curve has been divided into three sets of ordinates because the *magnitude* of $\sin 3\theta$ is repeated every $2\pi/3 = 120\,^\circ$.

$\theta\,^\circ$	y	$\theta\,^\circ$	y	$\theta\,^\circ$	y	$Y_3 = \Sigma y$	$\sin 3\theta$	$Y_3 \sin 3\theta$
10	4.65	130	3.00	250	0.30	7.95	0.500	1.930
20	5.80	140	3.42	260	−0.60	8.62	0.866	6.815
30	6.28	150	4.15	270	−1.15	9.28	1.000	5.330
40	6.30	160	4.10	280	−1.80	8.60	0.866	
50	6.00	170	3.66	290	−1.98	7.68	0.500	
60	5.42	180	2.78	300	−2.00	6.20	0.0	
70	4.86	190	2.30	310	−1.81	5.35	−0.500	
80	4.10	200	1.95	320	−1.78	4.27	−0.866	
90	3.55	210	1.70	330	−1.30	3.95	−1.000	
100	3.00	220	1.68	340	0.40	5.08	−0.866	
110	2.82	230	1.50	350	2.10	6.42	−0.500	
120	2.75	240	1.00	360	3.00	6.75	0.0	

To see how Eq. 5 works, consider that the three values under y have been added to provide $Y_3 = \Sigma y$. Then note that, ignoring zeros, the same magnitude of sine appears four times. So only three numbers appear in the last column, *e.g.*, the first is computed as follows.

$$0.500(7.95 + 7.68 - 5.35 - 6.42) = 1.930$$

Then

$$\sum_{k=1}^{36} Y_k \sin k \left(\frac{2\pi}{36}\right) = 1.930 + 6.815 + 5.330 = 14.075$$

And

$$a_3 = \frac{2}{12 \times 3} \times 14.075 = 0.782$$

Just so you won't have *too* much work to do and to give a second sample calculation using Eq. 6, let's calculate b_4. We'll need four columns.

$\theta\,^\circ$	y	$\theta\,^\circ$	y	$\theta\,^\circ$	y	$\theta\,^\circ$	y
10	4.65	100	3.00	190	2.30	280	−1.80
20	5.80	110	2.82	200	1.95	290	−1.98
30	6.28	120	2.75	210	1.70	300	−2.00
40	6.30	130	3.00	220	1.68	310	−1.81
50	6.00	140	3.42	230	1.50	320	−1.78
60	5.42	150	4.15	240	1.00	330	−1.30
70	4.86	160	4.10	250	0.30	340	0.40
80	4.10	170	3.66	260	−0.60	350	2.10
90	3.55	180	2.78	270	−1.15	360	3.00

Now for the last three columns.

$Y_4 = \Sigma y$	$\cos 4\theta$		$Y_4 \cos 4\theta$
8.15	0.7660	$(8.15 + 9.26)0.7660 =$	13.33606
8.59	0.1736	$(8.59 + 9.66)0.1736 =$	3.16850
8.73	−0.5000	$-(8.73 + 9.27)0.5000 =$	−13.36500
9.17	−0.9397	$-(9.17 + 9.14)0.9397 =$	−17.20590
9.14	−0.9397		
9.27	−0.5000		
9.66	0.1736		
9.26	0.7660		
8.18	1.0000	$(8.18)1 =$	8.18000

Then

$$\sum_{k=1}^{36} Y_k \cos k \left(\frac{2\pi}{36}\right) = -5.88634$$

And

$$b_4 = \frac{2}{9 \times 4} \times (-5.88643) = -0.327$$

Go ahead and finish the job.

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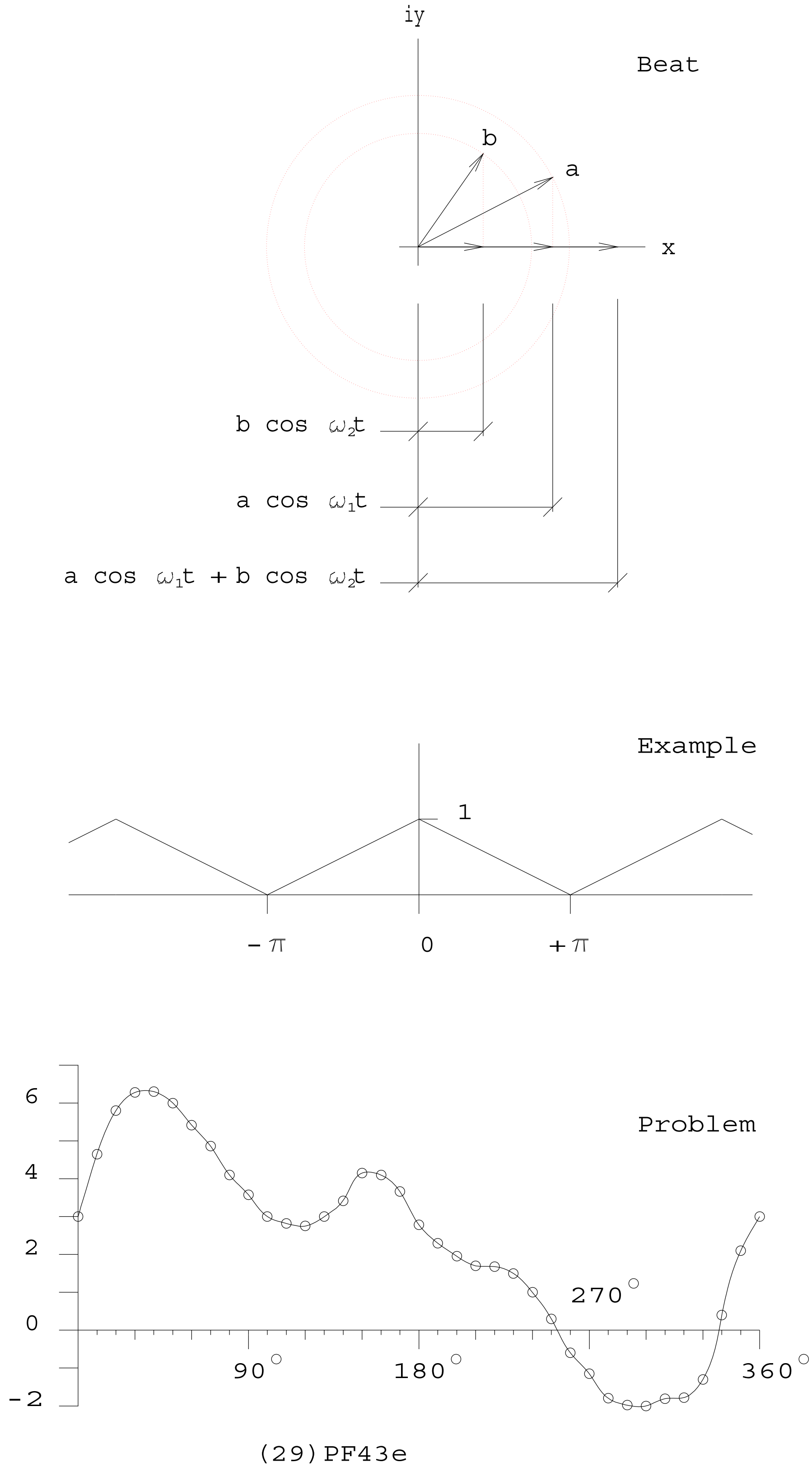

Figure 1: Beat and Analytical and Numerical Examples of Synthesis