Questions

1. What are geometry and blending matrices for Catmull-Rom spline?

2. The blending functions define a 4-vector of degree 3 polynomials in t:

$$\mathbf{B}\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}.$$

   Let the "vanilla version" be the $\mathbf{B}$ matrix used at the start of lecture 10. The blending function should be what, at $t = 0, 1, 2, 3$?

3. (a) Given three points $(x(0), y(0))$, $(x(1), y(1))$, $(x(2), y(2))$ in the (x, y) plane, show how to compute a 2×3 coefficient matrix $\mathbf{C}$ such that

$$\mathbf{q}(t) \;=\; \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \;=\; \mathbf{C}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

   and the curve $\mathbf{q}(t)$ passes through the three given points at $t = 0, 1, 2$. Note that the curve $\mathbf{q}(t)$ is *quadratic*, rather than cubic.

   (b) Give a formula for the tangent vector $\mathbf{q}'(t)$ to this curve (not necessily of unit length).

   (c) Suppose we are given only two points $\mathbf{q}(0)$ and $\mathbf{q}(1)$ in the plane. How could one define a *quadratic* curve through these points by choosing a suitable tangent vector, or a set of tangent vectors?

4. How would you obtain a degree 4 polynomial

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

   with given values $p(0), p(1), p(2), p(3), p(4)$?

Answers

1. We can write a Hermite geometry matrix and re-express it in terms of the points themselves rather than the tangents:

$$\begin{aligned} & \left[\ \mathbf{p}(i) \quad \mathbf{p}(i+1) \quad \mathbf{p}'(i) \quad \mathbf{p}'(i+1)\ \right] \\ = \quad & \left[\ \mathbf{p}(i-1) \quad \mathbf{p}(i) \quad \mathbf{p}(i+1) \quad \mathbf{p}(i+2)\ \right] \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

The first line above is $\mathbf{G}_{Hermite}$ for the segment from $\mathbf{p}(i)$ to $\mathbf{p}(i+1)$ and so we can substitute this into $\mathbf{G}_{Hermite}\mathbf{B}_{Hermite}$ to get

$$\mathbf{G}_{Hermite}\mathbf{B}_{Hermite} = \mathbf{G} \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \mathbf{B}_{Hermite} \ = \ \mathbf{G}\mathbf{B}_{Catmull-Rom}$$

where $\mathbf{B}_{Catmull-Rom}$ is *defined* to be the product of the matrix in the middle and $\mathbf{B}_{Hermite}$.

2. For $t = 0, 1, 2, 3$, the blending function should be

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The reason is that the curve is supposed to take exactly the given points at these t values.

3. (a)

$$\mathbf{C}_{3\times 2} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}$$

$$[\mathbf{q}(0) \ \ \mathbf{q}(1) \ \ \mathbf{q}(2)] = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

so

$$[\mathbf{q}(0) \ \ \mathbf{q}(1) \ \ \mathbf{q}(2)] \begin{bmatrix} 0 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}$$

(b)

$$\mathbf{q}'(t) = \mathbf{C} \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix}$$

(c) We can define a unique curve through two given 2D points by using the tangent vector of only one of the points, say $\mathbf{q}'(0)$, as follows:

$$\mathbf{q}(t) = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

and so

$$[\mathbf{q}(0) \;\; \mathbf{q}(1) \;\; \mathbf{q}'(0)] = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

so

$$[\mathbf{q}(0) \;\; \mathbf{q}(1) \;\; \mathbf{q}'(0)] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}$$

Thus we would define the curve by:

$$\mathbf{q}(t) = [\mathbf{q}(0) \;\; \mathbf{q}(1) \;\; \mathbf{q}'(0)] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

4.

$$\begin{bmatrix} p(0) & p(1) & p(2) & p(3) & p(4) \end{bmatrix} = \begin{bmatrix} a & b & c & d & e \end{bmatrix} \begin{bmatrix} 0 & 1 & 16 & 81 & 256 \\ 0 & 1 & 8 & 27 & 64 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Solve for the coefficients by multiplying by the inverse of the matrix on the right side.