Quantized and Sampled Control of Linear Second Order Systems

Hannah Michalska and Vincent Hayward

Abstract— **Continuous systems are today controlled digitally. It is therefore necessary to consider the effects of quantization and sampling. We show that any second-order LTI system can be controlled exactly by fixed quantized feedback independently from the resolution of the sensors. When sampling is considered, only practical stabilization can be achieved and the size of the limit cycle depends on the sampling rate.**

I. INTRODUCTION

Sampling and quantization effects have a special importance in control engineering since, today, analog control is employed only exceptionally. Since the very beginnings of digital control technology, research was concerned with the effects of sampling [1], [2], and quantization [3], [4]. Traditionally, the effects of quantization were approached by considered quantization errors as disturbances, for example, characterized as uniformly distributed noise [4]. The latter approach is advocated in contemporary texts on sampled-data systems [5].

Prior attempts to analyze the properties of quantized feeback systems include those of Delchamps [6], who found that systems with quantized feedback cannot be stabilized in the traditional sense. However, under the special assumption that the quantization step can be made arbitrarily small, Brockett and Liberzon showed that asymptotic stability can still be achieved [7]. When variable-rate sampling instants can be made to coincide in time with quantization, other convergence results were shown by Kofman [8]. Logarithmic quantization was considered by Elia and Mitter to show that a form of optimal control can still be achieved together with practical stabilization [9]. These, and other works, which cannot be surveyed here, often assume that the quantization step can be adjusted. In practice, however, fixed quantizers are used in most engineering systems (position encoders, analog-to-digital converters) where a fixed precision is designed to reflect the desired operating resolution.

In this article, we proceed with an exact analysis of fixed and uniformly quantized control of linear second-order systems.

II. PD-CONTROL

Second-order systems are important because the results obtained with these systems can be extended to other single-input, single-output (SISO), linear-time-invariant (LTI) systems. Generic second-order systems represent the dominant dynamics of many devices in mechanical and electrical applications. Among all possible compensators, proportional-derivative (PD) controllers are frequently employed in tracking and regulation for their robustness properties and because PD control coincides with a generic state-feedback control of a second-order system for which the poles can be placed arbitrarily.

With reference to the origin, the dynamics of a second-order LTI system with PD control can be written as

$$\ddot{x} = a_2\dot{x} + a_1 x + e, \quad e = -k_1' x - k_2'\dot{x}, \quad k_1', k_2' > 0. \quad (1)$$

If the states are accessible through sensors (that is, a velocity sensor is also available) and if the measurements are quantized and normalized to discrete levels, then (1) becomes

$$\ddot{x} = a_2\dot{x} + a_1 x - k_2(\lfloor \dot{x} \rfloor + \tfrac{1}{2}) - k_1(\lfloor x \rfloor + \tfrac{1}{2}). \quad (2)$$

Here, $\lfloor \varsigma \rfloor + \frac{1}{2}$ represents the action of the quantizer on a continuous signal, ς. The floor function returns the largest integer not exceeding ς, but since ς represents a signed quantity, $\frac{1}{2}$ has to be added to ensure symmetry of the quantization error with respect to zero. Most analog-to-digital converters, "mid-rise" types, are designed to behave this way so that ς and $-\varsigma$ have the same magnitude after quantization. When the origin is arbitrary, as in the case of incremental motion encoders, the origin can still be unambiguously put in correspondence between the continuous domain, $\varsigma \in \mathbb{R}$, and the quantized domain, $(\lfloor \varsigma \rfloor + \frac{1}{2}) - \frac{1}{2} \in \mathbb{Z}$.

III. EFFECT OF QUANTIZATION

A. Equilibrium of a Quantized Feedback System

We must first define an appropriate notion of equilibrium for systems such as (2). The right-hand-side of (2), evaluated at $x = 0$, is equal to $-\frac{1}{2}(k_1 + k_2)$, $k_1, k_2 > 0$. Hence, the origin is not a stationary point in an ordinary sense. In state space form, the motion of system (2) is governed by the discontinuous vector field $\boldsymbol{v}(\boldsymbol{x}) = [v_1(\boldsymbol{x}), v_2(\boldsymbol{x})]^\top \in \mathbb{R}^2$, $\boldsymbol{x} = [x_1, x_2]^\top \in \mathbb{R}^2$, where $x_1 \triangleq x$ and $x_2 \triangleq \dot{x}$,

$$\begin{aligned} v_1(\boldsymbol{x}) &= x_2, \\ v_2(\boldsymbol{x}) &= a_2 x_2 + a_1 x_1 - k_2(\lfloor x_2 \rfloor + \tfrac{1}{2}) - k_1(\lfloor x_1 \rfloor + \tfrac{1}{2}). \end{aligned} \quad (3)$$

The points of discontinuity in (3) are all the points in the set $D \triangleq \{x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}\}$. For further analysis, it is convenient to partition the state space into disjoint "tiles" within which the system vector field $\boldsymbol{v}$ is continuous,

$$S^{i,j} \triangleq \{x_1 \in]i, i+1[, x_2 \in]j, j+1[\}, \quad i, j \in \mathbb{Z}.$$

The closures of these sets are

$$\bar{S}^{i,j} \triangleq \{x_1 \in [i, i+1], x_2 \in [j, j+1]\}, \quad i, j \in \mathbb{Z}.$$

This work was supported by Natural Sciences and Engineering Council of Canada (NSERC) and the Université Pierre et Marie Curie (UPMC).

Hannah Michalska is with McGill Univ., Dept. of Electrical and Computer Engineering, Montréal, Canada, hannah.michalska@mcgill.ca

Vincent Hayward is with UPMC Univ Paris 06, Institut des Systèmes Intelligents et de Robotique, France, vincent.hayward@isir.fr

ISBN 978-963-311-369-1

It is convenient to introduce

$$\boldsymbol{v}^{i,j} : \boldsymbol{x} \mapsto \left[v_1^{i,j}(\boldsymbol{x}), v_2^{i,j}(\boldsymbol{x})\right]^\top,$$
$$\bar{\boldsymbol{v}}^{i,j} : \boldsymbol{x} \mapsto \left[\bar{v}_1^{i,j}(\boldsymbol{x}), \bar{v}_2^{i,j}(\boldsymbol{x})\right]^\top,$$

where a $\boldsymbol{v}^{i,j}$ is the restriction of $\boldsymbol{v}$ to the tile $S^{i,j}$, and $\bar{\boldsymbol{v}}^{i,j}$ is the continuous extension of the $\boldsymbol{v}^{i,j}$ to $\bar{S}^{i,j}$.

An approach to modeling systems such as $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ is to consider them as governed by differential inclusions [10], rather than by differential equations. These are written,

$$\dot{\boldsymbol{x}} \in F(\boldsymbol{x}). \tag{4}$$

The set-valued mapping $F : \boldsymbol{x} \mapsto F(\boldsymbol{x}) \subset \mathbb{R}^2, \forall \boldsymbol{x} \in \mathbb{R}^2$, is related to the piecewise continuous vector field $\boldsymbol{v}(\boldsymbol{x})$ as follows. When $\boldsymbol{x} \in S^{i,j}$ the RHS of (4) reduces to a singleton, $F(\boldsymbol{x}) = \{\boldsymbol{v}^{i,j}(\boldsymbol{x})\}$. For points on the boundaries of $\bar{S}^{i,j}$, $\boldsymbol{x} \in \partial \bar{S}^{i,j}$, for some $i, j \in \mathbb{Z}$, the set $F(\boldsymbol{x})$ is determined as the closed convex-hull of all the limit points $\boldsymbol{v}^*$ of the vector field $\boldsymbol{v}$ on the domain $S = \bigcup S^{i,j}$, where

$$\boldsymbol{v}^* = \lim_{\xi \to \boldsymbol{x}, \xi \in S} \boldsymbol{v}(\xi). \tag{5}$$

A solution to (4) is defined as an absolutely continuous function $\boldsymbol{x} : t \mapsto \boldsymbol{x}(t) \in \mathbb{R}^2$ such that $\dot{\boldsymbol{x}}(t) \in F(\boldsymbol{x}(t))$ almost everywhere. Then, see [10],

Definition 1: A point, $\boldsymbol{x}_E$, is a generalized stationary point of a quantized system such as (2) iff $\mathbf{0} \in F(\boldsymbol{x}_E)$.

With this definition, the constant trajectory $\boldsymbol{x}(t) = \boldsymbol{x}_E$ satisfies the differential inclusion (4). In particular, the origin is a generalized stationary point of (2) since $F(\mathbf{0}) = \frac{1}{2}\,\mathrm{co}\{[-k_1, -k_2]^\top, [k_1, -k_2]^\top, [k_1, k_2]^\top, [-k_1, k_2]^\top\} \ni \mathbf{0}$.

Proposition 1 (Unicity): A sufficient condition for the origin to be the sole stationary point of System (2) is that

$$k_1 - k_2 > 2\max(0, a_1). \tag{6}$$

Proof: First, note that $\forall \boldsymbol{x}$, $|x_2| > 0$, $\mathbf{0} \notin F(\boldsymbol{x})$, since any $\boldsymbol{v}$ has the form $[x_2, \cdot]^\top \neq \mathbf{0}$. We show now that the points of the form $[n + r, 0]^\top$, $r \in]0, 1[$, $n \in \mathbb{Z}$ and the points of the form $[n, 0]^\top$, $n \in \mathbb{Z} - \{0\}$ are not stationary. First, let $r > 0$. Then $F(\boldsymbol{x}) = \mathrm{co}\left\{\left[0, \bar{v}_2^{n,0}\right]^\top, \left[0, \bar{v}_2^{n,-1}\right]^\top\right\}$ with

$$\bar{v}_2^{n,0} = (a_1 - k_1)n + a_1 r - \tfrac{1}{2}k_1 - \tfrac{1}{2}k_2, \tag{7}$$
$$\bar{v}_2^{n,-1} = (a_1 - k_1)n + a_1 r - \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2. \tag{8}$$

We show that $\mathbf{0} \notin F([n+r, 0]^\top)$ by demonstrating that $\bar{v}_2^{n,0}$ and $\bar{v}_2^{n,-1}$ always have the same sign.

From (6), $a_1 - k_1 < 0$. From (7)-(8), $\bar{v}_2^{n,0}$ and $\bar{v}_2^{n,-1}$ have same signs $\forall n \in \mathbb{N} - \{0\}$, provided that both quantities are negative when $n = 0$. In the latter case, again by (6),

$$\bar{v}_2^{0,0} = a_1 r - \tfrac{1}{2}k_1 - \tfrac{1}{2}k_2 < 0,$$
$$\bar{v}_2^{0,-1} = a_1 r - \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2 < 0.$$

If now $n = -1$, then from (6),

$$\bar{v}_2^{-1,0} = -a_1 + k_1 + a_1 r - \tfrac{1}{2}k_1 - \tfrac{1}{2}k_2 > 0,$$
$$\bar{v}_2^{-1,-1} = -a_1 + k_1 + a_1 r - \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2 > 0.$$

As $a_1 - k_1 < 0$, one can see that $\bar{v}_2^{n,0} > 0$ and $\bar{v}_2^{n,-1} > 0$ when $-n \in \mathbb{N} - \{0, 1\}$, so $\mathbf{0} \notin F(\boldsymbol{x})$ for $-n \in \mathbb{N} - \{0\}$. Now, suppose that $r = 0$, then

$$F(\boldsymbol{x}) = \mathrm{co}\left\{\left[0, \bar{v}_2^{n-1,0}\right]^\top, \left[0, \bar{v}_2^{n-1,-1}\right]^\top, \left[0, \bar{v}_2^{n,0}\right]^\top, \left[0, \bar{v}_2^{n,-1}\right]^\top\right\},$$

for $n \in \mathbb{Z} - \{0\}$. When $n \geq 1$, then $[0, \bar{v}_2^{n-1,0}]^\top$, $[0, \bar{v}_2^{n-1,-1}]^\top$, $[0, \bar{v}_2^{n,0}]^\top$, $[0, \bar{v}_2^{n,-1}]^\top$ are all negative from (7)-(8). On the other hand, if $n \leq 1$ these quantities are all positive. Hence, $\mathbf{0} \notin F([n, 0]^\top)$, for all $n \in \mathbb{Z} - \{0\}$. ∎

Corollary 1: The sole constant trajectory of system (2) is $\boldsymbol{x}(t) = \mathbf{0}$, for all t.

B. Stability of a 2nd Order System with Quantized Feedback

At this point it is noted that, provided that Condition (6) holds, the formalism of differential inclusions is no longer needed since, away from the origin no ambiguity arises as to how the system traverses the discontinuities of the defining vector field. From the proof of Proposition 1, it can be seen that the trajectories of the system form a correct phase portrait that encircles the origin in the clockwise direction.

Under Condition (6) further analysis can hence be conducted by employing a standard system representation as in (2) in the region $R_D \triangleq \mathbb{R}^2 - \{\mathbf{0}\}$. The main stability result is easier to derive by first demonstrating that the closed-loop quantized system is input-output passive.

Proposition 2 (Passivity): A sufficient condition for the second-order system with quantized PD feedback (2) to be input-output passive with respect to the output $y = \dot{x}$ and the dissipation rate $\dot{x}u$ is that the gains satisfy

$$k_2 > 4\max\{0, a_2\}, \text{ and } k_1 > k_2 + 2\max\{0, a_1\}. \tag{9}$$

Proof: On R_D, consider the augmented system

$$\ddot{x} = a_2\dot{x} + a_1 x - k_2(\lfloor \dot{x} \rfloor + \tfrac{1}{2}) - k_1(\lfloor x \rfloor + \tfrac{1}{2}) + u, \tag{10}$$

and the associated input-output product

$$\dot{x}u = \dot{x}\ddot{x} - a_1\dot{x}x - a_2\dot{x}\dot{x} + \dot{x}k_1(\lfloor x \rfloor + \tfrac{1}{2}) + \dot{x}k_2(\lfloor \dot{x} \rfloor + \tfrac{1}{2}). \tag{11}$$

Integrating (11) over an arbitrary interval $[t_1, t_2]$ yields

$$\begin{aligned}\int_{t_1}^{t_2} \dot{x}u\,\mathrm{d}\tau &= \int_{t_1}^{t_2} \dot{x}\ddot{x}\,\mathrm{d}\tau - a_1\int_{t_1}^{t_2} \dot{x}x\,\mathrm{d}\tau - a_2\int_{t_1}^{t_2} \dot{x}\dot{x}\,\mathrm{d}\tau \\ &\quad + k_1\int_{t_1}^{t_2} \dot{x}(\lfloor x \rfloor + \tfrac{1}{2})\,\mathrm{d}\tau + k_2\int_{t_1}^{t_2} \dot{x}(\lfloor \dot{x} \rfloor + \tfrac{1}{2})\,\mathrm{d}\tau, \\ &= \tfrac{1}{2}[\dot{x}^2(t_2) - \dot{x}^2(t_1)] - a_1\tfrac{1}{2}[x^2(t_2) - x^2(t_1)] \\ &\quad + k_1\int_{t_1}^{t_2} \dot{x}(\lfloor x \rfloor + \tfrac{1}{2})\,\mathrm{d}\tau - a_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau + k_2\int_{t_1}^{t_2} \dot{x}(\lfloor \dot{x} \rfloor + \tfrac{1}{2})\,\mathrm{d}\tau.\end{aligned}$$

It is now convenient to introduce the quantization error function, $\xi \mapsto \Gamma(\xi) \triangleq \xi - \lfloor \xi \rfloor$. Then,

$$\int_{t_1}^{t_2} \dot{x}(\lfloor x \rfloor + \tfrac{1}{2})\,\mathrm{d}\tau = \int_{t_1}^{t_2} \dot{x}(x - \Gamma(x) + \tfrac{1}{2})\,\mathrm{d}\tau.$$

It is seen that

$$\int_{t_1}^{t_2} \dot{x}\Gamma(x)\,\mathrm{d}\tau = \int_{x(t_1)}^{x(t_2)} \Gamma(\zeta)\,\mathrm{d}\zeta = \int_0^{x(t_2)} \Gamma(\zeta)\,\mathrm{d}\zeta - \int_0^{x(t_1)} \Gamma(\zeta)\,\mathrm{d}\zeta$$
$$= \tfrac{1}{2}\left[\lfloor x(t_2)\rfloor + \Gamma^2(x(t_2)) - \lfloor x(t_1)\rfloor - \Gamma^2(x(t_1))\right]. \quad (12)$$

Hence,

$$\int_{t_1}^{t_2} \dot{x}(\lfloor x\rfloor + \tfrac{1}{2})\,\mathrm{d}\tau = \tfrac{1}{2}x^2(t_2) - \tfrac{1}{2}x^2(t_1)$$
$$+\tfrac{1}{2}\left[\Gamma(x(t_2)) - \Gamma^2(x(t_2))\right] - \tfrac{1}{2}\left[\Gamma(x(t_1)) - \Gamma^2(x(t_1))\right].$$

Now we show that a positive lower bound exists for the term

$$P(\dot{x}) \triangleq -a_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau + k_2\int_{t_1}^{t_2} \dot{x}(\lfloor \dot{x}\rfloor + \tfrac{1}{2})\,\mathrm{d}\tau$$

on the set R_D. WLOG, we may assume that t_1, t_2 are such that $\forall\tau \in [t_1, t_2]$ either

(a) $\dot{x}(\tau) = +n + r(t),\ n \in \mathbb{N} + \{0\}$, or
(b) $\dot{x}(\tau) = -n + r(t),\ n \in \mathbb{N}$.

Case (a): The value of term $P(\dot{x})$, for $r \in]0, 1[$, is

$$P(\dot{x}) = \int_{t_1}^{t_2} [-a_2(n + r(\tau))^2 + k_2(n + \tfrac{1}{2})(n + r(\tau))]\,\mathrm{d}\tau.$$

If $n = 0$, then $\dot{x}(\tau) = r(\tau) \in]0, 1[$ and

$$P(\dot{x}) = \int_{t_1}^{t_2} (-a_2\dot{x}^2 + \tfrac{1}{2}k_2\dot{x})\,\mathrm{d}\tau.$$

Choosing $k_2 \geq 4\max\{0, a_2\}$ and because $\dot{x} \in]0, 1[$,

$$P(\dot{x}) \geq \int_{t_1}^{t_2} (-a_2\dot{x}^2 + \tfrac{1}{2}k_2\dot{x}^2)\,\mathrm{d}\tau \geq \frac{1}{4}k_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau.$$

Now, if $n \in \mathbb{N}$,

$$\begin{aligned}P(\dot{x}) &= \int_{t_1}^{t_2} \big[-(a_2 - k_2)n^2 - 2a_2nr - a_2r^2 \\ &\qquad + k_2nr + \tfrac{1}{2}k_2n + \tfrac{1}{2}k_2r\big]\,\mathrm{d}\tau \\ &\geq -(a_2 - k_2)n^2(t_2 - t_1) + \int_{t_1}^{t_2} (-2a_2 + k_2)nr\,\mathrm{d}\tau \\ &\quad + \tfrac{1}{2}\int_{t_1}^{t_2} (-2a_2r + k_2)r\,\mathrm{d}\tau + \tfrac{1}{2}k_2n(t_2 - t_1) \\ &\geq -(a_2 - k_2)n^2(t_2 - t_1) + \tfrac{1}{2}k_2(t_2 - t_1).\end{aligned}$$

Case (b): $n \neq 0,\ r \in]0, 1[$,

$$\begin{aligned}P(\dot{x}) &= \int_{t_1}^{t_2} \big[-a_2(-n + r)^2 + k_2(-n + \tfrac{1}{2})(-n + r)\big]\,\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} \big[-a_2(-n + r) + k_2(-n + \tfrac{1}{2})\big](-n + r)\,\mathrm{d}\tau.\end{aligned}$$

If $n = 1$ then $\dot{x} = -1 + r \in]-1, 0[$, so $-\dot{x} \in]0, 1[$ and

$$\begin{aligned}P(\dot{x}) &= -a_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau + \tfrac{1}{2}k_2\int_{t_1}^{t_2} (-\dot{x})\,\mathrm{d}\tau \\ &\geq -a_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau + \tfrac{1}{2}k_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau \\ &\geq (-a_2 + \tfrac{1}{2}k_2)\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau \geq \frac{1}{4}k_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau.\end{aligned}$$

When $n \in \mathbb{N}, n \geq 2,\ r \in]0, 1[$,

$$-a_2(-n+r) + k_2(-n+\tfrac{1}{2}) < 0, \Rightarrow (a_2 - k_2)n - a_2r + \tfrac{1}{2}k_2 < 0.$$

Provided that $k_2 \geq 4\max\{0, a_2\}$ then

$$(a_2 - k_2)n - a_2r + \tfrac{1}{2}k_2 \leq -\frac{1}{2}k_2n.$$

Finally, the bound for case (b) is

$$\begin{aligned}P(\dot{x}) &= \int_{t_1}^{t_2} \big[-a_2(-n + r) + k_2(-n + \tfrac{1}{2})\big](-n + r)\,\mathrm{d}\tau \\ &\geq \tfrac{1}{2}k_2n(n - 1)(t_2 - t_1). \quad (13)\end{aligned}$$

Combining (12)-(13) gives

$$P(\dot{x}) \geq \begin{cases} \min\{\tfrac{1}{2}k_2; -(a_2 - k_2)\}n^2(t_2 - t_1), \\ \quad \text{if } |\dot{x}(\tau)| \in [n, n+1[, n \in \mathbb{N}, \forall\tau \in [t_1, t_2[, \\ \tfrac{1}{4}k_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau, \text{ if } |\dot{x}(\tau)| \in]0, 1[, \forall\tau \in [t_1, t_2[. \end{cases} \quad (14)$$

It follows that

$$\begin{aligned}\int_{t_1}^{t_2} \dot{x}u\,\mathrm{d}\tau =& \tfrac{1}{2}[\dot{x}^2(t_2) - \dot{x}^2(t_1)] - a_1\tfrac{1}{2}[x^2(t_2) - x^2(t_1)] \\ &+k_1\tfrac{1}{2}[x^2(t_2) - x^2(t_1)] + k_1\tfrac{1}{2}\left[\Gamma(x(t_2)) - \Gamma^2(x(t_2))\right] \\ &-k_1\tfrac{1}{2}\left[\Gamma(x(t_1)) - \Gamma^2(x(t_1))\right] \\ &-a_2\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}\tau + k_2\int_{t_1}^{t_2} (\lfloor \dot{x}\rfloor + \tfrac{1}{2})\dot{x}\,\mathrm{d}\tau, \\ =& \tfrac{1}{2}[\dot{x}^2(t_2) - \dot{x}^2(t_1)] - \tfrac{1}{2}(a_1 - k_1)[x^2(t_2) - x^2(t_1)] \\ &+ \tfrac{1}{2}k_1[\Gamma(x(t_2)) - \Gamma^2(x(t_2))] \\ &- \tfrac{1}{2}k_1[\Gamma(x(t_1)) - \Gamma^2(x(t_1))] + P(\dot{x}) \geq 0, \quad (15)\end{aligned}$$

which proves input-output passivity of (2) under conditions (9) with the storage function

$$S(x, \dot{x}) \triangleq \tfrac{1}{2}\left[\dot{x}^2 + (k_1 - a_1)x^2 + k_1(\Gamma(x) - \Gamma^2(x))\right]. \quad (16)$$

This storage function is well defined as it is positive definite since, for all x, $\Gamma(x) \in [0, 1[$ so $\Gamma(x) - \Gamma^2(x) \geq 0$. Also, S is locally Lipschitz continuous. The latter is established by showing Lipschitz continuity of the term $\Gamma(\cdot) - \Gamma^2(\cdot)$ at points $x \in \mathbb{Z}$. Let $x = n$ for any $n \in \mathbb{Z}$ and consider a neighbourhood $]n - \epsilon, n + \epsilon[, \epsilon \in]0, \tfrac{1}{2}[$ together with two points $n - \epsilon_1, n + \epsilon_2 \in]n - \epsilon, n + \epsilon[$. We note that

$$\lfloor n - \epsilon_1\rfloor = n - 1 \text{ so } n - \epsilon_1 = n - 1 + \Gamma(n - \epsilon_1),$$
$$\lfloor n + \epsilon_2\rfloor = n \text{ so } n + \epsilon_2 = n + \Gamma(n + \epsilon_2),$$

i.e. $\epsilon_1 = 1 - \Gamma(n - \epsilon_1)$ and i.e. $\epsilon_2 = \Gamma(n + \epsilon_2)$. Thus

$$\begin{aligned}&|\Gamma(n + \epsilon_2) - \Gamma^2(n + \epsilon_2) - \Gamma(n - \epsilon_1) + \Gamma^2(n - \epsilon_1)|, \\ \leq&|\Gamma(n + \epsilon_2) - \Gamma^2(n + \epsilon_2)| + |\Gamma(n - \epsilon_1) - \Gamma^2(n - \epsilon_1)|, \\ \leq&(1 - \epsilon_2)\epsilon_2 + (1 - \epsilon_1)\epsilon_1 \leq \epsilon_2 + \epsilon_1.\end{aligned}$$

Substituting back for $x = n$ shows that for any $\epsilon_1, \epsilon_2 < \epsilon$

$$\begin{aligned}&|\Gamma(x + \epsilon_2) - \Gamma^2(x + \epsilon_2) - \Gamma(x - \epsilon_1) + \Gamma^2(x - \epsilon_1)| \\ \leq&|(x + \epsilon_2) - (x + \epsilon_1)|,\end{aligned}$$

demonstrating Lipschitz continuity of $\Gamma(\cdot) - \Gamma^2(\cdot)$ with a Lipschitz constant equal to one. ■

Passivity entails asymptotic stability of the closed loop quantized system as shown below.

Theorem 1 (Stability): A second-order system with quantized PD feedback (2) is globally asymptotically stable if the gains satisfy Conditions (9).

Proof: Setting $u = 0$ in (15) and adopting the storage function (16) shows that

$$0 = S(\dot{x}(t_2), x(t_2)) - S(\dot{x}(t_1), x(t_1)) + P(\dot{x}),$$

so that on R_D,

$$S(\dot{x}(t_2), x(t_2)) - S(\dot{x}(t_1), x(t_1)) \leq -P(\dot{x}) < 0. \quad (17)$$

Since the storage function is proper, i.e., $S(x, \dot{x}) \to \infty$ as $\|[x, \dot{x}]\| \to \infty$, the level sets of S are compact and are invariant because S is decreasing monotonically along the trajectories of (2). Hence, the system is stable.

Since t_1, t_2 are arbitrary, it follows from (17) and the properties of the storage function, along with the lower bound in (14) and continuity of $\dot{x}$, that the trajectories of (2) approach the largest invariant set contained in the set

$$Z \triangleq \{(x, \dot{x}) \in \mathbb{R}^2 \,|\, P(\dot{x}) = 0\} = \{(x, \dot{x}) \in \mathbb{R}^2 \,|\, \dot{x} = 0\}.$$

We will show that the largest invariant set contained in Z is the constant, zero trajectory. Suppose that $x(\bar{t}) \neq 0$ for some $\bar{t} \in]t_1, t_2[$, but that $y(t) = \dot{x}(t) \equiv 0$ for all $t \in]t_1, t_2[$. Then, $\ddot{x}(t) \equiv 0$ for all $t \in]t_1, t_2[$ which implies that $x(t) = x(\bar{t}) \neq 0$ for all $t \in]t_1, t_2[$, i.e. $x(\bar{t})$ must be a stationary point of (2). This contradicts the result of Proposition 1 which states that the only stationary point is $x(\bar{t}) = 0$.

Hence, $Z = \{0\}$ and every trajectory of (2) approaches the origin. This proves asymptotic stability of the system. ■

IV. Illustrative Example

The stabilization result of Theorem 1 is somewhat counter-intuitive as one could think that a quantized feedback can only provide for practical stabilization (to a one-quantum neighborhood of the origin). As its stands, the result establishes asymptotic stabilization *regardless of the size of the quantum*. A simple example illustrates this phenomenon.

Consider the quantized PD control of a double integrator, i.e. assume that $a_1 = a_2 = 0$ in (2). Let the initial condition be $[x_1(0), x_2(0)] = [0, c]$ where $c > 0$ is such that $(0, c) \in \{[x_1, x_2] \,|\, S(x_1, x_2) < 1\}$, (less than one quantum away from $\mathbf{0}$). Referring to Fig. 1, quadrant by quadrant, the solution of (2) is a concatenation of portions of trajectories,

$$\mathrm{I}\begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = -\frac{1}{2}k_1 - \frac{1}{2}k_2,\end{cases} \quad \mathrm{II}\begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = -\frac{1}{2}k_1 + \frac{1}{2}k_2,\end{cases}$$

$$\mathrm{III}\begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = +\frac{1}{2}k_1 + \frac{1}{2}k_2,\end{cases} \quad \mathrm{IV}\begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = +\frac{1}{2}k_1 - \frac{1}{2}k_2.\end{cases}$$

For simplicity, denote $a \triangleq \frac{1}{2}(k_1 + k_2)$, and $b \triangleq \frac{1}{2}(k_1 - k_2)$ and assume that $b < a$ as guaranteed by Condition (6) of Proposition 1. Note that all four subsystems are unstable. In each quadrant, the fragments of the system trajectory emanating from the initial condition $(0, c)$ are easily computed. The

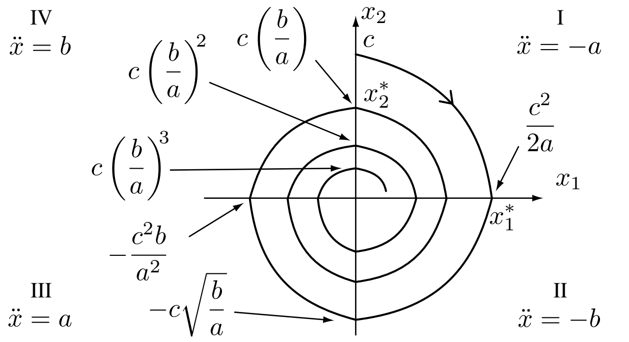

Fig. 1. Phase portrait of the double integrator with quantized PD controller.

points $[x_1^*, 0], [0, x_2^*]$ are the points of "scheduled switches" corresponding to switching instants t^*. Also, $c < \sqrt{2}$ as implied by $S(0, c) < 1$, such that in all cases $S(x_1^*, 0) < 1$ and $S(0, x_2^*) < 1$, as expected.

$$\mathrm{I}\begin{cases}x_2 = -at + c,\\ x_1 = -\frac{1}{2}at^2 + ct,\end{cases} \quad \mathrm{II}\begin{cases}x_2 = -bt,\\ x_1 = -\frac{1}{2}bt^2 + \frac{1}{2}\frac{c^2}{a},\end{cases}$$

$$x_2 = 0 \Rightarrow t^* = \frac{c}{a}, \quad x_1 = 0 \Rightarrow t^* = \frac{c}{\sqrt{ab}},$$

$$x_1^* = \frac{c^2}{2a} > 0. \quad x_2^* = -c\sqrt{\frac{b}{a}} < 0.$$

$$\mathrm{III}\begin{cases}x_2 = at - c\sqrt{\frac{b}{a}},\\ x_1 = \frac{1}{2}at^2 - c\sqrt{\frac{b}{a}}t,\end{cases} \quad \mathrm{IV}\begin{cases}x_2 = bt,\\ x_1 = \frac{1}{2}bt^2 - \frac{1}{2}\frac{c^2 b}{a^2},\end{cases}$$

$$x_2 = 0 \Rightarrow t^* = \frac{c}{a}\sqrt{\frac{b}{a}} \quad x_1 = 0 \Rightarrow t^* = \frac{c}{a},$$

$$x_1^* = -\frac{c^2 b}{2a^2} < 0. \quad x_2^* = \frac{cb}{a} < c.$$

The concatenated flow of the system is a "contraction" since $b/a < 1$. The trajectory of the system winds around the origin and passes through points $(0, c(\frac{b}{a})^n)$, $\forall n \in \mathbb{N}$. Figure 1 illustrates how the quantized PD controller applied to the double integrator causes the trajectory to adopt a phase portrait that resembles an ordinary stable focus. A system with quantized feedback can be yet another example of a stable switched system arising from a family of unstable members.

V. The Effect of Sampling

Actual digital controllers measure the output at discrete instants of time. Given a sampling period, T, the system with quantized and sampled control (2) becomes, $x_k = x(kT)$,

$$\ddot{x} = a_2\dot{x} + a_1 x - k_2(\lfloor \dot{x}_k \rfloor + \tfrac{1}{2}) - k_1(\lfloor x_k \rfloor + \tfrac{1}{2}), \quad (18)$$
$$\triangleq v_2(x, \dot{x}),\ \forall t \in [kT, (k+1)T[,\ \forall k \in \mathbb{N}.$$

The stationary points of (18) are still derived according to Definition 1 where the set-valued mapping $F : \boldsymbol{x} \mapsto F(\boldsymbol{x}) \subset \mathbb{R}^2, \forall \boldsymbol{x} \in \mathbb{R}^2$, is now calculated by taking limits as $\boldsymbol{x}_k \to \boldsymbol{x}$, i.e., the limit points $\boldsymbol{v}^*$ of $\boldsymbol{v}$ on the domain $S = \bigcup S^{i,j}$, when $\boldsymbol{x} \in \partial S^{i,j}$, are now assumed to be computed as

$$\boldsymbol{v}^* = \lim_{\boldsymbol{x}_k \to \boldsymbol{x}, \boldsymbol{x}_k \in S} \boldsymbol{v}(\boldsymbol{x}_k) \quad (19)$$

which replaces (5). With this modification the stationary points of the quantized and sampled system remain the same, that is, we can show what follows.

Proposition 3: A sufficient condition for the origin to be the sole stationary point of system (18) is given by (6).

Proof: As in the proof of Proposition 1, for all $\boldsymbol{x}$ such that $|x_2| > 0$, $\mathbf{0} \notin F(\boldsymbol{x})$, since any $\boldsymbol{v} \in F(\boldsymbol{x})$ has the form $\boldsymbol{v} = [x_2, \cdot]^\top \neq \mathbf{0}$. Adopting (19) to calculate the set-valued mapping $\boldsymbol{x} \mapsto F(\boldsymbol{x})$, it can be verified that the vector fields $\bar{\boldsymbol{v}}^{i,j} \in F(\boldsymbol{x}); i = 0, -1, n, n-1; j = 0, -1$ are the same as those found in the proof of Proposition 1. Hence, points of the form $[0, n+r]^\top$, $r \in [0, 1[$, $n \in \mathbb{Z}$ and of the form $[0, n]^\top$, $n \in \mathbb{Z} - \{0\}$ are not stationary points of (18). ■

Although passivity of the system is lost due to sampling, and so is asymptotic stability, it is possible to show what constitutes a type of robustness result for the feedback controller which employs only quantization. The system (18) is still practically stable in the following sense.

Theorem 2: Suppose that the gains of the feedback controller in (18) satisfy, as before,

$$k_2 > 4\max\{0, a_2\}, \quad k_1 > k_2 + 2\max\{0, a_1\}. \tag{20}$$

Then the quantized and sampled feedback control system (18) is practically stable in the sense that for every pair of level sets $S_\epsilon, S_\gamma, \gamma > \epsilon > 0$ of the storage function,

$$S_\epsilon \triangleq \{(x_1, x_2) \mid S(x_1, x_2) \leq \epsilon\},$$

there exists a sufficiently small sampling period T such that all system trajectories of (18) originating in S_γ reach S_ϵ in finite time and remain in S_ϵ thereafter.

Proof: For any given $\gamma > 0$, let $K_\gamma > 0$ be a bound for the RHS of (18) on the compact level set S_γ, i.e.

$$|v_2(x, \dot{x})| \leq K_\gamma \text{ , and also, } |\dot{x}| \leq \sqrt{2\gamma}, \tag{21}$$

$\forall [x, \dot{x}], [x_k, \dot{x}_k] \in S_\gamma$, where, WLOG, $\gamma > 1, K_\gamma > 1$ and $\epsilon < \frac{1}{2}$. The sampled system is now

$$\begin{aligned} \ddot{x} = &\, a_2\dot{x} + a_1 x - k_2(\lfloor \dot{x} \rfloor + \tfrac{1}{2}) - k_1(\lfloor x \rfloor + \tfrac{1}{2}) \\ &+ k_2(\lfloor \dot{x} \rfloor - \lfloor \dot{x}_k \rfloor) + k_1(\lfloor x \rfloor - \lfloor x_k \rfloor), \\ &\forall t \in [kT, (k+1)T[, \ \forall k \in \mathbb{N}. \end{aligned} \tag{22}$$

Along the solutions of this system the dissipativity inequality no longer holds as, on R_D, it is replaced by

$$\begin{aligned} &S(\dot{x}(t_2), x(t_2)) - S(\dot{x}(t_1), x(t_1)) \\ &\leq -P(\dot{x}) + k_2 \int_{t_1}^{t_2} |(\lfloor \dot{x} \rfloor - \lfloor \dot{x}(kT) \rfloor)\dot{x}| \,\mathrm{d}\tau \\ &\quad + k_1 \int_{t_1}^{t_2} |(\lfloor x \rfloor - \lfloor x(kT) \rfloor)\dot{x}| \,\mathrm{d}\tau. \end{aligned} \tag{23}$$

Now, let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of instants $t_{i+1} = t_i + \Delta t$, $\Delta t = \frac{h}{4K_\gamma\sqrt{\gamma}} < \frac{1}{2}$, for some h to be selected later. From the Mean Value Theorem,

$$|\dot{x}(t_{i+1}) - \dot{x}(t_i)| = |\ddot{x}(t_i + \theta)|\Delta t < h, \quad \theta \in [0, \Delta t] \tag{24}$$
$$|x(t_{i+1}) - x(t_i)| = |\dot{x}(t_i + \theta)|h \leq \sqrt{2\gamma}\Delta t < h, \quad \forall i \in \mathbb{N},$$

whenever $[x(t_i), \dot{x}(t_i)]^\top, [x(t_{i+1}), \dot{x}(t_{i+1})]^\top \in S_\gamma$. Note that

$$\lfloor \dot{x}(\tau) \rfloor - \lfloor \dot{x}_k \rfloor \neq 0 \text{ or } \lfloor x(\tau) \rfloor - \lfloor x_k \rfloor \neq 0,$$

only when $[x(\tau), \dot{x}(\tau)]^\top \in S^{n_1,n_2}$ and $[x(kT), \dot{x}(kT)]^\top \in S^{n_3,n_4}$ for different tiles of the state space. Now suppose that the trajectory of (22) satisfies, $[x(\tau), \dot{x}(\tau)]^\top \in S^C_{\epsilon/4} \cap S_\gamma$ for $\tau \in [t_i, t_{i+1}]$, where $S^C_{\epsilon/4}$ denotes the complement of the level set $S_{\epsilon/4}$. By virtue of (24) and since the origin is the sole stationary point of (22), there exists a constant $\bar{h}$ determining Δt such that $\lfloor x(\tau) \rfloor$ and $\lfloor \dot{x}(\tau) \rfloor$ change value at most once in the interval $[t_i, t_{i+1}]$ and such that $[x(\tau), \dot{x}(\tau)]^\top \in S_\epsilon, \forall \tau \in [t_i, t_{i+1}]$ whenever $[x(t_i), \dot{x}(t_i)]^\top \in S_{\epsilon/2}$. The value of h decreases to zero as $\epsilon \to 0$.

For the value of $\bar{h}$ selected as above, let $T = \frac{\Delta t}{m}$ which implies that sampling occurs at each time instant $t_i, i \in \mathbb{N}$, m to be selected later. The positive terms on the RHS of (23) can be bounded as follows,

$$\begin{aligned} &k_2 \int_{t_i}^{t_{i+1}} |(\lfloor \dot{x} \rfloor - \lfloor \dot{x}(kT) \rfloor)\dot{x}| \,\mathrm{d}\tau \\ &\leq k_2 \int_{t_i + jT}^{t_i + (j+1)T} |(\lfloor \dot{x} \rfloor - \lfloor \dot{x}(kT) \rfloor)| \, |\dot{x}| \,\mathrm{d}\tau \ \leq \ k_2 T \sqrt{2\gamma}, \end{aligned} \tag{25}$$

for some $j \in [0, m-1], j \in \mathbb{N}$. Similarly, on R_D,

$$k_1 \int_{t_i}^{t_{i+1}} |(\lfloor x \rfloor - \lfloor x_k \rfloor)\dot{x}| \,\mathrm{d}\tau \leq k_1 T \sqrt{2\gamma}.$$

We now need a lower bound for $P(\dot{x})$ in (23). From (14) and (20) it follows that

$$P(\dot{x}) \geq \begin{cases} \frac{1}{2}k_2(\sigma_2 - \sigma_1), \\ \text{if } |\dot{x}(\tau)| \in [n, n+1[, n \in \mathbb{N}, \forall \tau \in [\sigma_1, \sigma_2[, \\ \frac{1}{4}k_2 \int_{\sigma_1}^{\sigma_2} \dot{x}^2 \,\mathrm{d}\tau, \text{ if } |\dot{x}(\tau)| \in]0, 1[, \forall \tau \in [\sigma_1, \sigma_2[. \end{cases}$$

Using (24), it is easy to see that, if at t_i, $|\dot{x}(t_i)| \geq 1$ then $|\dot{x}(\tau)| \geq 1 - K_\gamma \Delta t > \frac{1}{2}$ for all $\tau \in [t_i, t_{i+1}]$ and

$$P(\dot{x}) \geq \tfrac{1}{2}k_2(\sigma - t_i) + \frac{1}{4}k_2 \int_\sigma^{t_{i+1}} \dot{x}^2 \,\mathrm{d}\tau \geq \frac{1}{8}k_2 \Delta t \tag{26}$$

where σ corresponds to the time instant at which $|\dot{x}(\sigma)| = 1$. Now, suppose that there exists a $\sigma \in [t_i, t_{i+1}]$ such that $\dot{x}(\sigma) = 0$. Under the Conditions (20), it follows from the proof of Proposition 1 that there exist constants $\psi > 0$ and $\delta \in]0, 1[$ such that, whenever $|\dot{x}| \leq \delta$,

$$a_1 x + a_2 \dot{x} - k_1(\lfloor x \rfloor + \tfrac{1}{2}) + \tfrac{1}{2}k_2 \leq -\psi \text{ if } x > 0,$$
$$a_1 x + a_2 \dot{x} - k_1(\lfloor x \rfloor + \tfrac{1}{2}) - \tfrac{1}{2}k_2 \geq \psi \text{ if } x < 0.$$

Also, it follows from (24) that for any $n \in \mathbb{N}$

$$|\dot{x}(\tau) - \dot{x}(kT)| \leq 2TK_\gamma, \tau \in](k-1)T, (k+1)T[,$$
$$|x(\tau) - x(kT)| \leq 2\sqrt{2\gamma}T, \ \tau \in](k-1)T, (k+1)T[.$$

whenever $[x(\tau), \dot{x}(\tau)]^\top, [x_k, \dot{x}_k]^\top \in S_\gamma$. Combining the above, and assuming that T is sufficiently small or, equivalently, that m is greater than some m^*, $\forall k \in \mathbb{N} : |\dot{x}_k| \leq \delta$

and $\forall \tau \in](k-1)T, (k+1)T[$ such that $|\dot{x}(\tau)| \leq \delta$

$$\begin{aligned}
&v_2(x(\tau), \dot{x}(\tau))\\
&\quad= a_1 x(\tau) + a_2\dot{x}(\tau) - k_1(\lfloor x(kT)\rfloor + \tfrac{1}{2}) + \tfrac{1}{2}k_2\\
&\quad= a_1 x(kT) + a_2\dot{x}(kT) - k_1(\lfloor x(kT)\rfloor + \tfrac{1}{2}) + \tfrac{1}{2}k_2\\
&\qquad + a_1(x(\tau) - x(kT)) + a_2(x(\tau) - x(kT))\\
&\quad\leq -\psi + |a_1| 2\sqrt{2\gamma}T + |a_2| 2TK_\gamma \leq -\tfrac{1}{2}\psi \text{ if } x(\tau) > 0,\\
&v_2(x(\tau), \dot{x}(\tau))\\
&\quad= a_1 x(\tau) + a_2\dot{x}(\tau) - k_1(\lfloor x(kT)\rfloor + \tfrac{1}{2}) - \tfrac{1}{2}k_2\\
&\quad= a_1 x(kT) + a_2\dot{x}(kT) - k_1(\lfloor x(kT)\rfloor + \tfrac{1}{2}) - \tfrac{1}{2}k_2\\
&\qquad + a_1(x(\tau) - x(kT)) + a_2(x(\tau) - x(kT))\\
&\quad\geq \psi - |a_1| 2\sqrt{2\gamma}T - |a_2| 2TK_\gamma \geq \tfrac{1}{2}\psi \text{ if } x(\tau) < 0.
\end{aligned}$$

The above implies that the system trajectory traverses the set $N_\delta \triangleq \{[x, \dot{x}]^\top \in \mathbb{R}^2 \mid |\dot{x}| \leq \delta\}$ in finite time; i.e. N_δ is not an invariant set of the system. From (21) the time t_R needed for the system trajectories to traverse N_δ is estimated as $t_R \geq \frac{2\delta}{K_\gamma} \triangleq t_B$. WLOG and not prejudicing the previous choice of $\bar{h}$, it can now be assumed that $\Delta t = 2t_B$. If there exists a $\sigma \in [t_k, t_{k+1}]$ such that $\dot{x}(\sigma) = 0$, then

$$\begin{aligned}
P(\dot{x}) &\geq \tfrac{1}{4}k_2 \int_{t_k}^{t_{k+1}} \dot{x}^2 \, \mathrm{d}\tau \geq \tfrac{1}{4}k_2 \int_0^{t_B} (\delta - K_\gamma \tau)^2 \, \mathrm{d}\tau,\\
&\geq \tfrac{1}{6}k_2 \frac{\delta^3}{K_\gamma} = \tfrac{1}{12}k_2\delta^2\Delta t. \qquad (27)
\end{aligned}$$

By virtue of (25), (26), (27), there exists a $c > 0$ such that

$$\begin{aligned}
&S(x(t_{k+1}), \dot{x}(x(t_{k+1})) - S(x(t_k), \dot{x}(t_k))\\
&\leq -\frac{1}{12}k_2\delta^2\Delta t + \sqrt{2\gamma}(k_1 + k_2)\frac{\Delta t}{m^*} \leq -c\Delta t, \qquad (28)\\
&\text{whenever } t_k \notin S_{\epsilon/4},
\end{aligned}$$

for a sufficiently large m^*. For such a value of m^*, select $T = \frac{\Delta t}{m^*}$. Since by (28) the storage function is guaranteed to decrease between time instants $t_k, k \in \mathbb{N}$, we claim that any trajectory starting from S_γ reaches $S_{\epsilon/2}$ in finite time. For if this were false, then we would have $S(x(t_i), \dot{x}(t_i)) \leq S(x(t_0), \dot{x}(t_0)) - cTi \to -\infty$ as $i \to \infty$ which contradicts the positive definiteness of S. Once $[x(t^*), \dot{x}(t^*)]^\top \in S_{\epsilon/2}$ then, by the choice of Δt, $[x(t^* + \Delta t), \dot{x}(t^* + \Delta t)]^\top \in S_\epsilon$ the storage function is again decreasing and the trajectory of the system cannot leave S_ϵ. ■

Introducing sampling destroys asymptotic stability. With the help of the example of the double integrator, we can show that, indeed, there does not exists a constant sampling time, $T > 0$, under which the quantized and sampled system trajectories converge to the origin. Assume that the initial condition of the system is, again, at $[0, c]^\top$. We can show that the largest delay that can be introduced in the switching while traversing the four quadrants of the phase plane must not exceed c/a if the system were to remain stable. Thus, a sampling period $T \leq 2c/a$ is required for stability.

A delay of c/a is equivalent to a system emanating from the initial condition $[0, c]^\top$ failing to switch when crossing the x_1-axis of the phase plane, i.e., the change in $\lfloor \dot{x} \rfloor$ is not detected at time $t^* = c/a$. The trajectory evolves through the four quadrants of the phase plane while the control switches only twice, as tabulated below:

t	$[x_1, x_2]^\top$	$\ddot{x}(\tau)$	τ
0	$[0, c]^\top$		
$\frac{c}{a}$	$\left[\frac{1}{2}\frac{c^2}{a}, 0\right]^\top$	$-a$	$[0, \frac{c}{a}[$
$\frac{2c}{a}$	$[0, -c]^\top$	$-a$	$[\frac{c}{a}, \frac{2c}{a}[$
$\frac{3c}{a}$	$\left[-\frac{1}{2}\frac{c^2}{a}, 0\right]^\top$	a	$[\frac{2c}{a}, \frac{3c}{a}[$
$\frac{4c}{a}$	$[0, c]^\top$	a	$[\frac{3c}{a}, \frac{4c}{a}[$

As can be seen, the system trajectory is periodic, demonstrating that using $T = \frac{2c}{a}$ produces trajectories that stay in the complement of the level set S_c. Asymptotic stabilization hence requires that $T \to 0$.

VI. Conclusion

Under the condition stipulated by Proposition 1, a continuous 2nd order system is asymptotically stabilized by quantized feedback for any quantum size, provided that the gains obey the passivity conditions of Proposition 2. This result is surprising since it is usually thought that stabilization is only possible within the resolution of the sensors. For set-point control, the system can be controlled to converge exactly to any quantization boundary. Considering quantization alone is akin to assuming that the sampling rate is sufficiently fast to ignore its effects, otherwise, only practical stabilization can be achieved and most systems will enter a limit cycle in the neighborhood of the origin.

This theory needs to be extended to the quantized and sampled control of any LTI system. We also plan to relate it to practical problems such as achieving accurate control in virtual reality haptic systems [11], as well as to lift the assumption that velocity sensors are available.

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