Time-Warped Bandlimited Signals: Sampling, Bandlimitedness, and Uniqueness of Representation

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1. INTRODUCTION

The ability to reconstruct a complex-valued signal on $\mathbb{R}$ from a sequence of sample values $\{f(t_n)\}_{n\in\mathbb{Z}}$ is desirable in a variety of engineering applications. While this problem is ill-posed in general, many reconstruction formulas of the form

$$f(t) = \sum_{n\in\mathbb{Z}} f(t_n)\varphi_n(t) \tag{1}$$

have been obtained for various restricted classes of functions.

It was observed in [1] that such a formula for reconstruction of functions from a given class $\mathcal{C}$ extends directly to a reconstruction formula for functions formed by composition of any $f \in \mathcal{C}$ with an invertible function $\gamma : \mathbb{R} \to \mathbb{R}$. Application of a coordinate transformation such as γ to the domain of a signal is commonly called "time-warping" in signal processing literature. Consequently, signals of this type have become known as "time-warped" signals.

Among the most important formulas of the type (1) are connected with reconstruction of bandlimited signals; i.e., functions having the form

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)e^{i\omega t}\, d\omega \tag{2}$$

where $\hat{f} \in L^2(\mathbb{R})$ and $0 < \Omega < \infty$. Motivated by their reconstructability from samples, this note presents some comments on the class $\mathcal{B} \circ \Gamma$ of time-warped bandlimited signals; i.e., functions of the form $f \circ \gamma$ with f belonging to the class $\mathcal{B}$ of bandlimited signals and $\gamma : \mathbb{R} \to \mathbb{R}$ belonging to a class Γ of continuous and invertible warping functions.

2. RESULTS

The perspective of Paley and Wiener [3] that it is natural to consider bandlimited functions on the complex domain is adopted in what follows. It thus becomes necessary to consider warping functions on $\mathbb{C}$ as well. Given a bandlimited function $f : \mathbb{R} \to \mathbb{C}$, denote by F the corresponding entire function with values defined by

$$F(z) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)e^{i\omega z}\, d\omega$$

Similarly, given $h \in \mathcal{B}$, denote by H the associated entire function. Define $\mathcal{G}$ to be the collection of all continuous functions $G : \mathbb{C} \to \mathbb{C}$ with restrictions γ to $\mathbb{R}$ that are real-valued and bijective. If $G \in \mathcal{G}$ then the corresponding $\gamma \in \Gamma$ is well defined. Thus, given bandlimited functions F and H on the complex domain, finding a $G \in \mathcal{G}$ such that $H = F \circ G$ ensures that there is some $\gamma \in \Gamma$ such that $h = f \circ \gamma$. Given $\gamma \in \Gamma$ such that $h = f \circ \gamma$, however, there is no *a priori* guarantee that any $G \in \mathcal{G}$ exists with the property that $H = F \circ G$. In this sense, considering complex warping functions in G is more restrictive than considering real-valued warping functions in Γ.

Theorem 1: If $f \in \mathcal{B}$ is not identically zero and $G \in \mathcal{G}$, then $H = F \circ G$ is bandlimited if and only if G is affine.

It is clear that $H = F \circ G$ will be bandlimited if G is affine. The proof of the "only if" part of this theorem is based on the growth properties of the entire functions F and H. Specifically, it relies on the following results.

Lemma: Suppose $G \in \mathcal{G}$, $f \in \mathcal{B}_\Omega$ is not identically zero, and $H = F \circ G$ is bandlimited, then G is entire.

Theorem 2 (from [4]): If F and G are entire and the order of $F \circ G$ is finite, then either (i) G is a polynomial and the order of F is finite, or (ii) G is a non-polynomial function of finite order and the order of F is zero.

Theorem 3 (based on results from [4]): If $f \in \tilde{\mathcal{B}}$ is not identically zero and G is a polynomial of degree $n > 1$, then the order of $H = F \circ G$ is greater than one.

The proof of Theorem 1 proceeds as follows. Assuming H is bandlimited, Theorem 1 establishes G is entire. Theorem 2 may be applied to show that G is a polynomial. Theorem 3 implies that the degree of G is either zero or one. If G were constant then H would be constant. Since $h \in \mathbf{L}^2$, it cannot be constant without being identically zero. Thus G is a polynomial of degree exactly one; i.e., $G(z) = az + b$ with $a \neq 0$. The condition that γ is real valued implies that a and b are real. Hence $\gamma(t) = at + b$ for real numbers a and b with $a \neq 0$.

3. DEMODULATION

Earlier work [2] has established that $\mathcal{B} \circ \Gamma$ contains all bandlimited functions and many nonbandlimited functions, but not all of $\mathbf{L}^2$. A remaining issue is that of *demodulation*: given $h \in \mathcal{B} \circ \Gamma$, can it be decomposed into a bandlimited function f and a bijective monotone time warping function γ?

If $h \in \mathcal{B} \circ \mathcal{G}$, then there are necessarily many ways to express h as a composition $f \circ \gamma$. Given any $\alpha > 0$, for example, define functions f_1 and γ_1 by $f_1(t) = f(\alpha t)$ and $\gamma_1(t) = \gamma(t/\alpha)$. Then $f_1 \in \mathcal{B}$, $\gamma_1 \in \mathcal{G}$, and $f_1 \circ \gamma_1 = f \circ \gamma = h$. This kind of representational ambiguity can be circumvented by stipulating that f have exactly unit bandwidth. In this case, the question of representational ambiguity may be addressed by a corollary to Theorem 1.

Corollary [of Theorem 1]: Suppose $h = f_1 \circ \gamma_1 = f_2 \circ \gamma_2$ with f_1 and f_2 having exactly unit bandwidth and $\gamma_1, \gamma_2 \in \mathcal{G}$. Then $f_1(t) = f_2(t-b)$ and $\gamma_1(t) = \gamma_2(t) + b$ for some real constant b and all $t \in \mathbb{R}$.

References

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