ON THE SAMPLING AND RECONSTRUCTION OF TIME-WARPED BANDLIMITED SIGNALS

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ABSTRACT

When a bandlimited signal is time warped by composition with a monotone function, the resulting signal is generally not bandlimited. Nevertheless, it has been observed that such signals can be reconstructed from sample sets of finite density. This paper examines the sampling and reconstruction problem for time-warped bandlimited signals and discusses its relationship to the more general sampling problem for non-bandlimited signals.

I. INTRODUCTION

Standard Shannon or Nyquist sampling theory forms a cornerstone of modern digital signal processing by providing a mechanism for sampling and reconstruction of bandlimited signals. In [1], it was observed that this theory extends to allow reconstruction of certain non-bandlimited signals from sets of non-uniformly spaced samples. This paper examines the space of signals to which this observation applies.

II. TIME-WARPED BANDLIMITED SIGNALS

In the following sections, $\mathcal{B}$ will denote the set of real-valued, finite-energy ($\mathbf{L}^2$) signals defined on $(-\infty,\infty)$ with spectra F that vanish outside the interval $[-\Omega,\Omega]$. Γ will represent the collection of all real-valued monotone functions on $(-\infty,\infty)$. Note that each $\gamma \in \Gamma$ has an inverse γ^{-1} with the property that $\gamma^{-1}(\gamma(t)) = \gamma(\gamma^{-1}(t)) = t$ for all $t \in (-\infty,\infty)$.

With this notation, the observation contained in [1] is as follows: If $f \in \mathcal{B}$ and $\gamma \in \Gamma$, then the time-warped signal $h = f \circ \gamma$ formed by composing f with γ [ie., the signal with values $h(t) = f(\gamma(t))$] can be reconstructed from samples $h_n \triangleq h(\gamma^{-1}(nT))$ where T denotes the Nyquist sampling interval for f. This reconstruction is possible because the samples $h_n = h(\gamma^{-1}(nT)) = f(\gamma(\gamma^{-1}(nT))) = f(nT)$ are in fact Nyquist samples of f. Thus f can be reconstructed from the samples h_n and then composed with γ to yield the desired signal h.

Denoting the collection of signals that can be formed by time warping of a bandlimited signal as $\mathcal{B} \circ \Gamma$, it is immediately apparent that $\mathcal{B} \subset \mathcal{B} \circ \Gamma$. In fact, it follows immediately from the shifting and scaling properties of the Fourier transform that time warping of a bandlimited signal by any *affine* function γ (ie., $\gamma(t) = at+b$ with $a \neq 0$) results in a bandlimited signal. The following section shows that $\mathcal{B} \circ \Gamma$ indeed contains many signals that are not bandlimited, but does not contain every $\mathbf{L}^2$ signal. The remaining sections of this paper deal with the problem of approximating an arbitrary $\mathbf{L}^2$ signal by a signal in $\mathcal{B} \circ \Gamma$ and the demodulation problem of approximately decomposing a given signal into the form $f \circ \gamma$ for some $f \in \mathcal{B}$ and $\gamma \in \Gamma$.

III. SOME PROPERTIES OF BANDLIMITED SIGNALS

There are numerous approaches to demonstrating that $\mathcal{B} \circ \Gamma$ contains signals that are not bandlimited. An example given in [1] shows conceptually that it is possible to produce a non-bandlimited signal by time warping a bandlimited one. Two more rigorous examples, following classical work in harmonic analysis, are based upon two properties of bandlimited functions.

The first of these properties is that a bandlimited function cannot approach zero as $t \to \infty$ faster than the function e^{-t} does. Mathematically, every bandlimited function f must satisfy

$$\limsup_{t\to\infty} |e^t f(t)| = \infty \tag{1}$$

This property, which may be thought of as a stronger form of the well known feature that a bandlimited function cannot be time limited, follows from a theorem due to Levinson (see [5] or [6], Theorem XXII) and motivated by the work of Paley and Wiener [7].

This property allows construction of simple examples of non-bandlimited functions in $\mathcal{B} \circ \Gamma$:

Example 1 *The function f with values*

$$f(t) = \frac{\sin(t)}{t} \tag{2}$$

is well known to be bandlimited. However $f(e^t)$ cannot be bandlimited because

$$\limsup_{t\to\infty} |e^t f(e^t)| = \limsup_{\tau\to\infty} |\tau f(\tau)| = \limsup_{\tau\to\infty} |\sin(\tau)| = 1 \tag{3}$$

A second property of bandlimited functions of the form

$$f(t) = \int_{-\Omega}^{\Omega} F(\omega) e^{\imath\omega t} d\omega \tag{4}$$

is that they are *entire* [8] if t is considered as a complex variable [9]. Because the only entire function that is zero on any interval of the real line is identically zero, a broad class of non-bandlimited functions in $\mathcal{B} \circ \Gamma$ is established:

Theorem 1 *If $f \in \mathcal{B}$ and γ is affine on any interval, then $f \circ \gamma \notin \mathcal{B}$ unless γ is affine on $(-\infty, \infty)$.*

Proof Suppose $\gamma(t) = at + b$ with $a \neq 0$ on some interval $\mathcal{I}$, but not on all of $(-\infty, \infty)$. If $g(t) = f(at + b)$ then $g \in \mathcal{B}$, but the signal $[f \circ \gamma] - g$ is not bandlimited because it is zero on $\mathcal{I}$. Therefore $g \notin \mathcal{B}$.

In particular, this theorem implies that a piecewise-linear time warping of a bandlimited function cannot be bandlimited. It also shows that signals in $\mathcal{B}$ are sensitive to arbitrarily small time warpings:

Example 2 *Define a monotone function γ by*

$$\gamma(t) = \begin{cases} t^{1+\epsilon} & 0 \leq t \leq 1 \\ t & \textit{otherwise} \end{cases} \tag{5}$$

Then $\sup|\gamma(t) - t|$ and $\int |\gamma(t) - t| dt$ can both be made arbitrarily small by choice of ϵ. But, if $\epsilon \neq 0$, $f \circ \gamma \notin \mathcal{B}$ for any $f \in \mathcal{B}$.

These properties suggest that, not only are there many cases where a time-warped bandlimited signal is not bandlimited, but that this is the typical situation. This is expressed in [2] as a conjecture:

Conjecture 1 *If $f \in \mathcal{B}$ and $\gamma \in \Gamma$, then $f \circ \gamma \in \mathcal{B}$ if and only if γ is affine.*

IV. TIME-WARPED BANDLIMITED SIGNALS IN $\mathbf{L}^2$

The previous two sections establish that a broad class of non-bandlimited signals can be reconstructed from sets of non-uniformly spaced samples having finite sample density [1]. However, the space $\mathcal{B} \circ \Gamma$ does not contain every $\mathbf{L}^2$ signal. The signal

$$g(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

cannot be expressed in the form $g = f \circ \gamma$ for $f \in \mathcal{B}$ and $\gamma \in \Gamma$, for example.

Approximation of $\mathbf{L}^2$ Signals

Although not every $g \in \mathbf{L}^2$ is in $\mathcal{B} \circ \Gamma$, it is possible to approximate every such g by a time-warped bandlimited function $h = f \circ \gamma$ so that the total-square error

$$||g - h||^2 \triangleq \int_{-\infty}^{\infty} |g(t) - h(t)|^2 dt \tag{7}$$

is arbitrarily small. That such an approximation is always possible follows from the Plancherel theorem [8] which shows that any $g \in \mathbf{L}^2$ can indeed be approximated by a bandlimited signal f. Therefore, any such g has an approximation as $f \circ \gamma$ where $f \in \mathcal{B}$ and $\gamma(t) = t$. This is equivalent to the statement that any $\mathbf{L}^2$ signal can be reconstructed with arbitrary accuracy from uniformly spaced samples by making the sampling rate sufficiently high.

Representational Ambiguity

Having shown that each $g \in \mathbf{L}^2$ can be approximated by a time-warped bandlimited signal, it is important to note that there will generally be several approximations in this form of equal quality. One reason that this is true is because every time-warped bandlimited signal has infinitely many equivalent representations of the form $f \circ \gamma$ with $f \in \mathcal{B}$ and $\gamma \in \Gamma$. Suppose, for example, γ_1 is affine and γ_2 is monotone, and denote $\gamma_3 = \gamma_1 \circ \gamma_2$. Then if $f \in \mathcal{B}$, the signal $g = f \circ \gamma_3$ may also be expressed as $g = \phi \circ \gamma_2$ where $\phi = f \circ \gamma_1 \in \mathcal{B}$.

This representational ambiguity may be eliminated by considering the collection $\mathcal{B}_{[0,1]}$ of all $f \in \mathcal{B}$ with Fourier transforms F having *support interval* $[0, 1]$ (see [8]). In other words, $\mathcal{B}_{[0,1]}$ consists of signals whose band is contained in the interval $0 \leq \omega \leq 1$ but is not contained in any smaller interval. Any $f \in \mathcal{B}$ can be expressed uniquely as the time warping of some $h \in \mathcal{B}_{[0,1]}$ by an affine function $\gamma(t) = at + b$ with $a > 0$. Furthermore, as argued in [2], if Conjecture 1 holds and $g = f_1 \circ \gamma_1 = f_2 \circ \gamma_2$ with $f_i \in \mathcal{B}_{[0,1]}$ and γ_i monotone increasing for $i = 1, 2$, then $f_1 \equiv f_2$ and $\gamma_1 \equiv \gamma_2$ (ie., this representation is unique). With this in mind, bandlimited signals f will be assumed to be in $\mathcal{B}_{[0,1]}$ and monotone functions will be assumed to be *increasing* for the remainder of this paper.

Approximation with Non-Affine Warpings

Even in the absence of the representational ambiguity discussed above, there will generally be several approximations of a signal in $\mathbf{L}^2$ by time-warped bandlimited signals that will yield total-square less than a given upper bound. Suppose, for example, $g = f_1 \circ \gamma_1$ is a non-bandlimited $\mathbf{L}^2$ signal formed by time warping a bandlimited signal. Then, as discussed above, g may be approximated with arbitrary accuracy by a bandlimited signal $h = f_2 \circ \gamma_2$ where γ_2 is affine. Thus, given any total-square error threshold $\epsilon > 0$, the representations $f_1 \circ \gamma_1$ and $f_2 \circ \gamma_2$ are both satisfactory approximations of g. There are also non-bandlimited approximations of g of the form $f \circ \gamma$ with $f \neq f_1$ and $\gamma \neq \gamma_1$ that satisfy the same total-square error bound.

Because the underlying purpose of the approximation is to sample g, a reasonable criterion for deciding among the numerous approximations of g that yield acceptable reconstruction errors is to choose one that minimizes the sample density. If a non-bandlimited signal g is approximated as $f \circ \gamma$ with γ affine, reconstruction error will generally go to zero only as the sampling rate approaches infinity. If g is actually a time-warped bandlimited signal, it is desirable to determine the functions f and γ of which g is composed because:

1. The representation, and the corresponding sampling and reconstruction processes, are exact; and
2. The number of samples required to achieve a reconstruction of acceptable accuracy may be smaller.

If $g \notin \mathcal{B} \circ \Gamma$, sample density is minimized by an approximation $f \circ \gamma$ with $f \in \mathcal{B}_{[0,1]}$ and

$$s(t) \triangleq |\gamma(t) - \gamma(-t)| \tag{8}$$

having *order* [8] as small as possible.

V. SAMPLING OF NON-BANDLIMITED SIGNALS

In [2], the process of time warping a bandlimited signal is likened to a *generalized phase modulation* in which the bandlimited signal f is regarded as a *carrier signal* and the time-warping function γ as *modulating function*. In these terms, the problem of decomposing an arbitrary $g \in \mathbf{L}^2$ into the form $f \circ \gamma$ may be regarded as a generalized demodulation problem. The sampling and reconstruction scheme discussed in the first section of this paper assumed that the modulating function γ was known. In practice, the modulating function is often not known and the sampling technique requires that *implicit* estimates of both f and γ be made using measurements involving only the values of g.

As discussed in the same reference, finding an exact demodulation for an arbitrary $\mathbf{L}^2$ signal is a difficult problem. It may be expressed as an ill-posed inverse problem that can be approached by regularization into the form of an energy functional minimization problem, but this approach offers little hope for practical applications.

On the other hand, it is relatively easy to devise *ad hoc* approaches for approximate demodulation. Suppose g is to be approximated by a function of the form $f \circ \gamma$ where f has unit bandwidth. If the bandwidth of g is estimated locally at each t to yield an estimate $B(t)$ of the "local bandwidth" of g [4], then γ may be estimated by observing that $\gamma'(t)$ should be proportional to $B(t)$. In practice, B may be obtained from f in a variety of ways – such as time-windowed spectral estimation. An appealing aspect of this perspective is that it satisfies the intuitive notion that the appropriate sampling rate for a signal with time-varying local bandwidth should vary in the same way that the local bandwidth does.

VI. CONCLUDING REMARKS

The authors believe that the approach discussed in this short paper provides significant potential for efficient sampling of certain non-bandlimited signals, particularly those that are "bursty" in nature. The current topics of emphasis in our continuing work in this area include:

1. Establishing the validity of the "folk theorem" of Conjecture 1. We are pursuing an approach that appears promising, involving the representation of bandlimited functions as everywhere-convergent power series at a single point in the time domain;

2. Developing and empirically evaluating practical demodulation and implicit sampling techniques; and

3. Formulating this perspective on sampling as part of a generalized theory of *run-length encoding* [3].

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