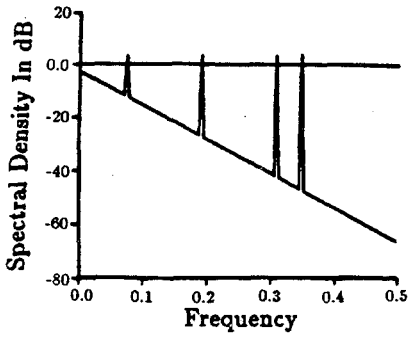

Fig. 2. The true spectrum.

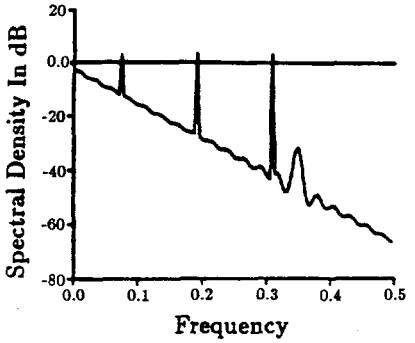

Fig. 3. The MCE spectrum based on 20 cepstral coefficients.

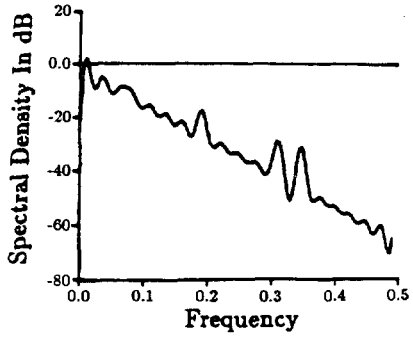

Fig. 4. The 20th-order AR spectrum.

that is much closer to the true spectrum than that produced by the autoregressive method.

VI. Conclusion

In conclusion, the MCE method proposed by Tzannes *et al.* can be derived without Lagrange multiplier formalism and can be solved without explicit solution of nonlinear equations.

References

[1] R. B. Blackman and J. W. Tukey, "The measurement of the power spectrum from the point of view of communication engineering," *Bell. Syst. Tech. J.*, vol. 33, pp. 185–282, 485–569, 1958.

[2] J. P. Burg, "Maximum entropy spectral analysis," in *Proc. 37th Meet. Soc. Explor. Geophys.*, Oklahoma City, OK, 1967.

[3] J. E. Shore, "Minimum cross-entropy spectral analysis," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, Apr. 1981.

[4] R. W. Johnson and J. E. Shore, "Minimum cross-entropy spectral analysis of multiple signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, June 1983.

[5] R. W. Johnson, J. E. Shore, and J. P. Burg, "Multisignal minimum cross-entropy spectrum analysis with weighted initial estimates," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 531–535, June 1984.

[6] R. W. Johnson, "Axiomatic characterization of the directed divergences and their linear combinations," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 709–716, Nov. 1979.

[7] J. E. Shore and R. W. Johnson, "Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 26–37, Jan. 1980.

[8] ——, "Properties of cross-entropy minimization," *IEEE Trans. Inform. Theory*, vol IT-27, pp. 472–482, July 1981.

[9] R. W. Johnson and J. E. Shore, "Which is the better entropy expression for speech processing: $-S \log S$ or $\log S$?" *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 129–136, Feb. 1984.

[10] R. Weinstock, *Calculus of Variations with Applications to Physics and Engineering*. New York: McGraw-Hill, 1952.

[11] A. Papoulis, "Maximum entropy and spectral estimation: A review," *IEEE Trans. Acoust, Speech, Signal Processing*, vol. ASSP-29, pp. 1176–1186, Dec. 1981.

Analysis of the Error in the Reconstruction of *N*-Dimensional Stochastic Processes

JAMES J. CLARK AND PETER D. LAWRENCE

***Abstract*—This correspondence derives a frequency domain expression for the error in the reconstruction of an *N*-dimensional stochastic process from its uniformly distributed samples when the reconstruction technique of Petersen and Middleton is used with an arbitrary reconstruction filter.**

Introduction

In this correspondence we present a derivation of the mean square, and average mean square error (averaged over a sample cell), of the reconstruction of a stochastic process from uniformly distributed samples of that process. The reconstruction method so analyzed is that of Petersen and Middleton [1], who derived an expression for the reconstruction error in the case of ideal filtering. We extend their analysis to the case of nonideal reconstruction filters, such as truncated ideal filters.

Definitions

In this section we present some definitions that will be used in the following.

N is the number of dimensions.

$$\int_X = \int_{x_1} \int_{x_2} * * * \int_{x_N=-\infty}^{\infty} \tag{1}$$

$$\int_\Omega = \int_{\omega_1} \int_{\omega_2} * * * \int_{\omega_N=-\infty}^{\infty} \tag{2}$$

$$\vec{x} = (x_1, x_2, \cdots, x_N) \tag{3}$$

$$\vec{\omega} = (\omega_1, \omega_2, \cdots, \omega_N). \tag{4}$$

Manuscript received February 3, 1986; revised April 21, 1986. This work was supported in part by the Joint Services Electronics Program at Harvard University and the Science Council of British Columbia.

J. J. Clark is with the Division of Applied Sciences, Harvard University, Cambridge, MA 02138.

P. D. Lawrence is with the Department of Electrical Engineering, University of British Columbia, Vancouver, B.C., Canada, V6T 1W5.

IEEE Log Number 8609637.

$\{f(\vec{x})\}$ is an ensemble of random functions and represents a wide-sense homogeneous ("stationary") stochastic process.

$f_R(\vec{x})$ is the reconstructed value of one of the random functions in the ensemble.

The ensemble average covariance is given by

$$K(\vec{z}) = E\{f(\vec{x})\, f(\vec{y})\} = K(\vec{x} - \vec{y}) \tag{5}$$

and has a Fourier transform, which exists everywhere except possibly at isolated delta function singularities:

$$\Phi(\vec{\omega}) = \int_X K(\vec{x})\, e^{-j\vec{\omega}\cdot\vec{x}}\, \vec{x}. \tag{6}$$

Reconstruction of N-Dimensional Stochastic Processes

The reconstruction method of Petersen and Middleton is summarized below in the form of a theorem (abstracted from [1]; this theorem is also to be found in [2]).

Theorem: Let the process $f(\vec{x})$ be sampled at points on the uniform sampling lattice defined by

$$\{\vec{x}_s\} = \{\vec{x} : \vec{x} = l_1\vec{v}_1 + l_2\vec{v}_2 + \cdots + l_N\vec{v}_N\} \tag{7}$$

where the l_k are integers and the $\vec{v}_k$ are independent vectors. Let the hypervolume of a basic sampling cell be denoted Q. Define the dual, frequency domain, sampling lattice to $\{\vec{x}_s\}$ by

$$\{\vec{\omega}_s\} = \{\vec{\omega} : \vec{\omega} = l_1\vec{u}_1 + l_2\vec{u}_2 + \cdots + l_N\vec{u}_N \tag{8}$$

where

$$\vec{v}_j \cdot \vec{u}_k = 2\pi\delta_{jk}; \qquad j, k = 1, 2, \cdots, N \tag{9}$$

and δ_{jk} is the Kronecker delta. Let us define the reconstruction of $f(\vec{x})$ from its samples $\{f(\vec{x}_s)\}$ to be as follows:

$$f_R(\vec{x}) = \sum_{\{\vec{x}_s\}} f(\vec{x}_s)\, g(\vec{x} - \vec{x}_s) \tag{10}$$

where $g(\vec{x})$ is the reconstruction filter, whose Fourier transform is $G(\vec{\omega})$.

If $\Phi(\vec{\omega})$ vanishes outside of the basic dual lattice cell (e.g., the hypervolume with sides $\vec{u}_k$, $k = 1, 2, \cdots, N$), and if $G(\vec{\omega})$ equals Q where $\Phi(\vec{\omega})$ is nonzero, is zero where the repetitive images $\Phi(\vec{\omega} + \vec{\omega}_s)$ are nonzero (for $\vec{\omega}_s \neq \vec{0}$), and is arbitrary everywhere else, then the mean square error of the reconstruction

$$\bar{e}^2(\vec{x}) = E\{[f(\vec{x}) - f_R(\vec{x})]^2\} \tag{11}$$

vanishes everywhere. □

Derivation of the Mean Square Error

If the reconstruction filter $G(\vec{\omega})$ does not satisfy the conditions outlined in the theorem, or if the stochastic process is not suitably bandlimited to the basic samping cell in the frequency domain, then the mean square error will not vanish for all $\vec{x}$.

The mean square error, for the most general case, can be written as

$$\bar{e}^2(\vec{x}) = K(\vec{0}) - 2\sum_{\{\vec{x}_s\}} K(\vec{x} - \vec{x}_s)\, g(\vec{x} - \vec{x}_s) + \sum_{\{\vec{x}_s\}}\sum_{\{\vec{x}_s\}} K(\vec{x}_{s1} - \vec{x}_{s2})\, g(\vec{x} - \vec{x}_{s1})\, g(\vec{x} - \vec{x}_{s2}). \tag{12}$$

Petersen and Middleton [1] derive the mean square error for the case of reconstruction with the ideal reconstruction filter which was defined in the theorem. Their result is

$$\bar{e}^2(\vec{x}) = K(\vec{0}) - \sum_{\{\vec{x}_s\}} K(\vec{x} - \vec{x}_s)\, g(\vec{x} - \vec{x}_s) = \frac{1}{(2\pi)^N}\int_\Omega \left[\Phi(\vec{\omega}) - \frac{G(\vec{\omega})}{Q}\sum_{\{\vec{\omega}_s\}} e^{-j\vec{x}\cdot\vec{\omega}_s}\,\Phi(\vec{\omega} + \vec{\omega}_s)\right] d\vec{\omega}. \tag{13}$$

This equation gives the mean square error (as a function of $\vec{x}$) in the reconstructed process for the case of an ideal filter and an arbitrary, not necessarily bandlimited, process. However, it is more often the case that the reconstruction filter is not the ideal one required by the theorem. We will now derive the mean square error for the case where both $g(\vec{x})$ and $f(\vec{x})$ are arbitrary.

It can be seen from (13) and the fact that

$$K(\vec{0}) = \frac{1}{(2\pi)^N}\int_\Omega \Phi(\vec{\omega})\, d\vec{\omega} \tag{14}$$

that the first two terms in the general expression (12) for the mean square error can be written

$$K(\vec{0}) - 2\sum_{\{x_s\}} K(\vec{x} - \vec{x}_s)\, g(\vec{x} - \vec{x}_s) = \frac{1}{(2\pi)^N}\int_\Omega \left[\Phi(\vec{\omega}) - \frac{2G(\vec{\omega})}{Q}\sum_{\{\omega_s\}} e^{-j\vec{x}\cdot\vec{\omega}_s}\,\Phi(\vec{\omega} + \vec{\omega}_s)\right] d\vec{\omega}. \tag{15}$$

Thus, in order to get a complete expression for the mean square error, we need to find an expression for the third term in (12). Let us call this term T for convenience.

We can expand T in terms of delta functions as follows, using the integral properties of the delta function:

$$T = \int_X\int_X K(\vec{r} - \vec{s})\, g(\vec{x} - \vec{r})\, g(\vec{x} - \vec{s}) \cdot \sum_{\{\vec{x}_s\}}\sum_{\{\vec{x}_s\}} \delta(\vec{r} - \vec{x}_{s1})\,\delta(\vec{s} - \vec{x}_{s2})\, d\vec{r}\, d\vec{s}. \tag{16}$$

It can be shown (using the result of [1, Append. A]) that

$$\sum_{\{\vec{x}_s\}}\sum_{\{\vec{x}_s\}} \delta(\vec{r} - \vec{x}_{s1})\,\delta(\vec{s} - \vec{x}_{s2}) = \frac{1}{Q^2}\sum_{\{\vec{\omega}_s\}}\sum_{\{\vec{\omega}_s\}} e^{j\vec{r}\cdot\vec{\omega}_{s1}}\, e^{j\vec{s}\cdot\vec{\omega}_{s2}}. \tag{17}$$

Thus, we can write

$$T = \frac{1}{Q^2}\sum_{\{\vec{\omega}_s\}}\sum_{\{\vec{\omega}_s\}}\int_X\int_X K(\vec{r} - \vec{s})\, g(\vec{x} - \vec{r})\, g(\vec{x} - \vec{s}) \cdot e^{j\vec{r}\cdot\vec{\omega}_{s1}}\, e^{j\vec{s}\cdot\vec{\omega}_{s2}}\, d\vec{r}\, d\vec{s}. \tag{18}$$

If we make the change of variables $\vec{y} = \vec{r} - \vec{x}$ and $\vec{z} = \vec{s} - \vec{x}$, and assume that $g(\vec{x})$ and $K(\vec{x})$ are even functions, we obtain

$$T = \alpha\int_X\int_X K(\vec{y} - \vec{z})\, g(\vec{y})\, g(\vec{z})\, e^{j\vec{y}\cdot\vec{\omega}_{s1}}\, e^{j\vec{z}\cdot\vec{\omega}_{s2}}\, d\vec{y}\, d\vec{z} \tag{19}$$

where we have defined, for simplicity,

$$\alpha = \frac{1}{Q^2}\sum_{\{\vec{\omega}_s\}}\sum_{\{\vec{\omega}_s\}} e^{j\vec{x}\cdot(\vec{\omega}_{s1} + \vec{\omega}_{s2})}. \tag{20}$$

Separating out the functions that depend only on $\vec{y}$ gives us

$$T = \alpha\int_X g(\vec{y})\, e^{j\vec{y}\cdot\vec{\omega}_{s1}}\left[\int_X K(\vec{y} - \vec{z})\, g(\vec{z})\, e^{j\vec{z}\cdot\vec{\omega}_{s2}}\, d\vec{z}\right] d\vec{y}. \tag{21}$$

The integral inside the brackets can be recognized as a convolution. Hence, we can write

$$T = \alpha\int_X g(\vec{y})\, e^{j\vec{y}\cdot\vec{\omega}_{s1}}\,[K(\vec{y}) \cdot g(\vec{y})\, e^{j\vec{y}\cdot\vec{\omega}_{s2}}]\, d\vec{y}. \tag{22}$$

Replacing the bracketed term by its Fourier transform representation gives

$$T = \frac{\alpha}{(2\pi)^N}\int_X g(\vec{y})\, e^{j\vec{y}\cdot\vec{\omega}_{s1}}\left[\int_\Omega \Phi(\vec{\omega})\, G(\vec{\omega} + \vec{\omega}_{s2})\, e^{+j\vec{y}\cdot\vec{\omega}}\, d\vec{\omega}\right] d\vec{y}. \tag{23}$$

Replacing $\vec{y}$ by $-\vec{y}$, assuming $g(\vec{y}) = g(-\vec{y})$, and rearranging gives us

$$T = \frac{\alpha}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega})\, G(\vec{\omega} + \vec{\omega}_{s2}) \left[\int_X g(\vec{y})\, e^{-j\vec{y}\cdot\vec{\omega}_{s1}}\, e^{-j\vec{y}\cdot\vec{\omega}}\, d\vec{y} \right] d\vec{\omega}. \tag{24}$$

Evaluation of the bracketed integral as a Fourier transform yields

$$T = \frac{\alpha}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega})\, G(\vec{\omega} + \vec{\omega}_{s2})\, G(\vec{\omega} - \vec{\omega}_{s1})\, d\vec{\omega}. \tag{25}$$

We can now write down the complete frequency domain expression for the mean square reconstruction error, valid for arbitrary $G(\vec{\omega})$ and $\Phi(\vec{\omega})$:

$$\bar{e}^2(\vec{x}) = \frac{1}{(2\pi)^N} \int_\Omega \left[\Phi(\vec{\omega}) - \frac{2G(\vec{\omega})}{Q} \sum_{\{\omega_s\}} e^{-j\vec{x}\cdot\vec{\omega}^s}\, \Phi(\vec{\omega} + \vec{\omega}_s) + \frac{G(\vec{\omega} - \vec{\omega}_{s1})\, G(\vec{\omega} + \vec{\omega}_{s2})}{Q^2} \cdot \sum_{\{\vec{\omega}_s\}} \sum_{\{\vec{\omega}_s\}} e^{+j\vec{x}\cdot(\vec{\omega}_{s1} + \vec{\omega}_{s2})}\, \Phi(\vec{\omega}) \right] d\vec{\omega}. \tag{26}$$

Derivation of the Average Mean Square Error

An often more useful error measure, and one that is more readily computed, is the mean square error averaged over a sampling cell. Let us call this measure E_{avg} and denote the spatial support of an elemental sampling cell by Γ. It can be seen that E_{avg} is given by

$$E_{\text{avg}} = K(\bar{0}) - \frac{1}{Q} \int_\Gamma \left[\sum_{\{\vec{x}_s\}} 2K(\vec{x} - \vec{x}_s)\, g(\vec{x} - \vec{x}_s) + \sum_{\{\vec{x}_s\}} \sum_{\{\vec{x}_s\}} K(\vec{x}_{s1} - \vec{x}_{s2})\, g(\vec{x} - \vec{x}_{s1})\, g(\vec{x} - \vec{x}_{s2}) \right] d\vec{x}. \tag{27}$$

Making a change of variables ($\vec{y} = \vec{x} - \vec{x}_s$) and noting that the summation of integrals over the elementary sampling cells Γ is the same as integrating over the entire space X allows us to write

$$\frac{1}{Q} \int_\Gamma \sum_{\{\vec{x}_s\}} 2K(\vec{x} - \vec{x}_s)\, g(\vec{x} - \vec{x}_s)\, d\vec{x} = \frac{2}{Q} \int_X K(\vec{x})\, g(\vec{x})\, d\vec{x}, \tag{28}$$

and hence,

$$E_{\text{avg}} = K(\bar{0}) - \frac{2}{Q} \int_X K(\vec{x})\, g(\vec{x})\, d\vec{x} + \frac{1}{Q} \int_\Gamma \cdot \sum_{\{\vec{x}_s\}} \sum_{\{\vec{x}_s\}} K(\vec{x}_{s1} - \vec{x}_{s2})\, g(\vec{x} - \vec{x}_{s1})\, g(\vec{x} - \vec{x}_{s2})\, d\vec{x}. \tag{29}$$

Let us define the function $r(\vec{x})$ as follows:

$$r(\vec{x}) = 1, \quad \vec{x} \in \Gamma$$
$$r(\vec{x}) = 0, \quad \text{otherwise}. \tag{30}$$

The third term of (29) can be seen to equivalent to

$$\frac{1}{Q} \int_X r(\vec{x}) \sum_{\{\vec{x}_s\}} \sum_{\{\vec{x}_s\}} K(\vec{x}_{s1} - \vec{x}_{s2})\, g(\vec{x} - \vec{x}_{s1})\, g(\vec{x} - \vec{x}_{s2})\, d\vec{x}. \tag{31}$$

Using the definition of T [i.e., the third term of (12)] and the expression for T given by (25), we can rewrite (31) as follows:

$$\frac{1}{(2\pi)^N Q^3} \sum_{\{\vec{\omega}_s\}} \sum_{\{\vec{\omega}_s\}} \int_X r(\vec{x})\, e^{j\vec{x}\cdot(\vec{\omega}_{s1} + \vec{\omega}_{s2})}\, d\vec{x} \cdot \int_\Omega \Phi(\vec{\omega})\, G(\vec{\omega} - \vec{\omega}_{s1})\, G(\vec{\omega} - \vec{\omega}_{s2})\, d\vec{\omega}. \tag{32}$$

Recognizing the first integral in the above expression as a Fourier transfrom allows us to rewrite this expression as

$$\frac{1}{(2\pi)^N Q^3} \sum_{\{\vec{\omega}_s\}} \sum_{\{\vec{\omega}_s\}} R(-(\vec{\omega}_{s1} + \vec{\omega}_{s2})) \cdot \int_\Omega \Phi(\vec{\omega})\, G(\vec{\omega} - \vec{\omega}_{s1})\, G(\vec{\omega} + \vec{\omega}_{s2})\, d\vec{\omega} \tag{33}$$

where $R(\vec{\omega})$ is the Fourier transform of $r(\vec{x})$. Petersen and Middleton [1, Append. D] show that

$$R(\vec{\omega}_s) = Q \quad \text{for} \quad \vec{\omega}_s \in \{\vec{\omega}_s\} = \vec{0}$$
$$= 0 \quad \text{for} \quad \vec{\omega}_s \in \{\vec{\omega}_s\} \neq \vec{0}. \tag{34}$$

This means that since $\vec{\omega}_{s1} + \vec{\omega}_{s2} \in \{\vec{\omega}_s\}$, $R(\vec{\omega}_{s1} + \vec{\omega}_{s2}) = Q$ if $\vec{\omega}_{s1} = -\vec{\omega}_{s2}$ and is zero otherwise. Thus, we can rewrite (33) as

$$\frac{1}{(2\pi)^N Q^2} \sum_{\{\vec{\omega}_s\}} \int_\Omega \Phi(\vec{\omega})\, G^2(\vec{\omega} + \vec{\omega}_s)\, d\vec{\omega}. \tag{35}$$

Combining this result with (14) and (29), we obtain

$$E_{\text{avg}} = \frac{1}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega}) \left[1 - \frac{1}{Q^2} \sum_{\{\vec{\omega}_s\}} G^2(\vec{\omega} + \vec{\omega}_s) \right] d\vec{\omega} - \frac{2}{Q} \int_X K(\vec{x})\, g(\vec{x})\, d\vec{x}. \tag{36}$$

We now will derive a frequency domain expression for $(2/Q) \int_X K(\vec{x})\, g(\vec{x})\, d\vec{x}$. First, we replace $K(\vec{x})\, g(\vec{x})$ by its Fourier transform representation:

$$\int_X K(\vec{x})\, g(\vec{x})\, d\vec{x} = \int_X \left[\frac{1}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega}) \cdot G(\vec{\omega})\, e^{j\vec{\omega}\cdot\vec{x}}\, d\vec{\omega} \right] d\vec{x} \tag{37}$$

where $\cdot$ denotes the convolution operation. We can rewrite this as

$$\int_\Omega \Phi(\vec{\omega}) \cdot G(\vec{\omega}) \left[\frac{1}{(2\pi)^N} \int_X e^{j\vec{\omega}\cdot\vec{x}}\, d\vec{x} \right] d\vec{\omega}. \tag{38}$$

Evaluation of this gives

$$\int_X K(\vec{x})\, g(\vec{x})\, d\vec{x} = \frac{1}{(2\pi)^N} \Phi(\vec{\omega}) \cdot G(\vec{\omega}) \Big|_{\vec{\omega} = \vec{0}}. \tag{39}$$

Writing out the convolution integral and setting $\vec{\omega} = \vec{0}$, we get [assuming $G(\vec{\omega}) = G(-\vec{\omega})$]

$$\int_X K(\vec{x})\, g(\vec{x})\, d\vec{x} = \frac{1}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega})\, G(\vec{\omega})\, d\vec{\omega}. \tag{40}$$

We can now write the complete frequency domain expression for the average mean square error. It is

$$E_{\text{avg}} = \frac{1}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega}) \left[1 - \frac{2}{Q} G(\vec{\omega}) + \frac{1}{Q^2} \sum_{\{\vec{\omega}_s\}} G^2(\vec{\omega} + \vec{\omega}_s) \right] d\vec{\omega}. \tag{41}$$

If $G(\vec{\omega})$ is the ideal reconstruction filter, then the expression for the average mean square error reduces to the following:

$$E_{\text{avg, ideal}} = \frac{1}{(2\pi)^N} \int_\Omega \Phi(\vec{\omega}) \left[1 - \frac{G(\vec{\omega})}{Q} \right] d\vec{\omega}. \tag{42}$$

This result was also obtained by Petersen and Middleton [1]. The main difference between the results for the nonideal and ideal reconstruction fitlers is that the average mean square error in the nonideal case is a function of the sample set, whereas in the ideal case the average mean square error is independent of the sample set.

Clark [3] provides examples of the computation of the average mean square error in two dimensions, for the case of Gaussian reconstruction filters. He also discusses extensions of the derivation given in this paper to the case of reconstruction from nonuniformly distributed sample sets.

Summary

We have presented a derivation of frequency domain expressions for the mean square error, and the mean square error averaged over a sampling cell, in the reconstruction of an N-dimensional stochastic process from its samples on a uniform grid. This extends the work of Petersen and Middleton [1] in that it allows for arbitrary (nonideal) reconstruction filters.

References

[1] D. P. Petersen and D. Middleton, "Sampling and reconstruction of wave-number limited functions in N-dimensional Euclidean spaces," *Inform. Contr.*, vol. 5, pp. 279-323, 1962.

[2] D. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[3] J. Clark, "Multi-resolution stereo vision with application to the automated measurement of logs," Ph.D. dissertation, Dep. Elec. Eng., Univ. British Columbia, Vancouver, B.C., Canada, Sept. 1985.

Comparison of Various Time Delay Estimation Methods by Computer Simulation

ANTONI FERTNER AND ANDERS SJÖLUND

***Abstract*—This correspondence provides qualitative estimates of the magnitude of the error in the measured delay time resulting from the error on the observed cross-correlation curve. The variances of five time delay estimators have been obtained by computer simulation to demonstrate the accuracy of all five methods. The comparison of the different correlation techniques shows that the average magnitude difference function gives results almost as accurate as direct correlation.**

I. Introduction

Correlation techniques have been used extensively in various scientific and technological fields for a number of years [1]-[7]. Time delay is a basic estimate in many applications. A common application comprises two spatially separated sensors which register the signal emanating from a remote source. The correlated signals are assumed to be bandlimited stationary Gaussian processes corrupted by noncross-correlating noise. The position of the peak in an observed cross-correlation curve is interpreted as the time delay estimate.

Because of practical interest, the implementation of different methods has been a research topic for a long time. Several methods exist for computing cross correlation from data which are related but not identical. However, their implementations for practical application have been limited by the high hardware costs. Thus, the choice of suitable methods compromising accuracy and economy requirements is of particular importance.

The purpose of this correspondence is to provide qualitative estimates of the magnitude of the error of the measured delay time. Computer simulation seems to be a particularly attractive method because it allows empirical comparison of the variance of the time delay estimator for all the methods under the same circumstances.

II. Definitions and Method of Comparison

Different time delay estimators have been proposed, discussed, and used in specific applications [7]-[10]. In this correspondence, we shall report the simulation studies concerning variance of the time delay estimators as a function of signal-to-noise ratio (SNR), number of samples (N), and quantization accuracy.

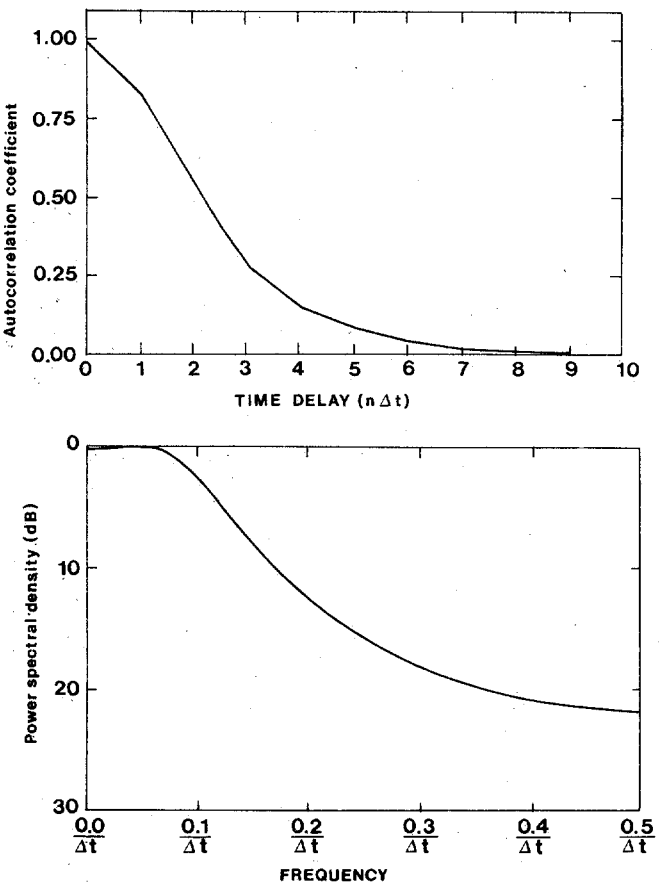

Fig. 1. Sampled autocorrelation function (ACF) and power spectral density (PSD) of the random time series $s(i\Delta t)$.

The computer experiment starts with the generation of the random sequence $s(i\Delta t)$ with autocorrelation function and spectral density shown in Fig. 1. The signal-plus-noise sequences $x_1(i\Delta t)$ and $x_2(i\Delta t)$ are formed by adding two independent Gaussian sequences $n_1(i\Delta t)$ and $n_2(i\Delta t)$ to $s(i\Delta t)$ and its delayed version

$$x_1(i\Delta t) = s(i\Delta t) + n_1(i\Delta t)$$
$$x_2(i\Delta t) = s(i\Delta t - D) + n_2(i\Delta t) \quad (1)$$

where D is the time delay and Δt denotes the sampling interval. For each realization of $x_1(i\Delta t)$ and $x_2(i\Delta t)$ the location of the correlation peak was determined by the following methods.

1) Direct correlation [10]:

$$R_{DC}(\tau) = \frac{1}{N} \sum x_1(i\Delta t) \cdot x_2(i\Delta t + \tau). \quad (2)$$

2) Hybrid-sign correlation [10]:

$$R_{HS}(\tau) = \frac{1}{N} \sum x_1(i\Delta t) \cdot \text{sign}\,(x_2(i\Delta t + \tau)). \quad (3)$$

3) Polarity-coincidence correlation [8], [10]:

$$R_{PC}(\tau) = \frac{1}{N} \sum \text{sign}\,(x_1(i\Delta t)) \cdot \text{sign}\,(x_2(i\Delta t + \tau)). \quad (4)$$

4) Average magnitude difference function [9]:

$$R_{AMDF}(\tau) = \frac{1}{N} \sum |x_1(i\Delta t) - x_2(i\Delta t + \tau)|. \quad (5)$$

Manuscript received August 12, 1985; revised February 13, 1986.

The authors are with the Institute of Microwave Technology, P.O. Box 70033, S-100 44 Stockholm, Sweden.

IEEE Log Number 8609042.