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Errata
Fundamentals of Signals & Systems
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with impulse response $u[n]$ to the signal $h[n]$. Hence, the step response of a discrete time LTI system is just the running sum of its impulse response:

$$s[n] = \sum_{-\infty}^{n} h[k]. \tag{2.41}$$

Conversely, the impulse response of the system is the output of the firs difference system with the step response as the input:

$$h[n] = s[n] - s[n-1]. \tag{2.42}$$

For a continuous-time system with impulse response $h(t)$, the step response $s(t) = u(t) * h(t) = h(t) * u(t)$. This convolution can also be interpreted as the r sponse of the integrator system with impulse response $u(t)$ to the signal $h(t)$. Agai the step response of a continuous-time LTI system is just the running integral of i impulse response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau. \tag{2.4}$$

Conversely, the first-order differentation system (the inverse system of the i tegrator), applied to the step response, yields the impulse response of the syste

$$h(t) = \frac{d}{dt}s(t). \tag{2.4}$$

Example 2.14: The impulse response and the step response of the *RC* circuit of Figure 2.33 are shown in Figure 2.34.

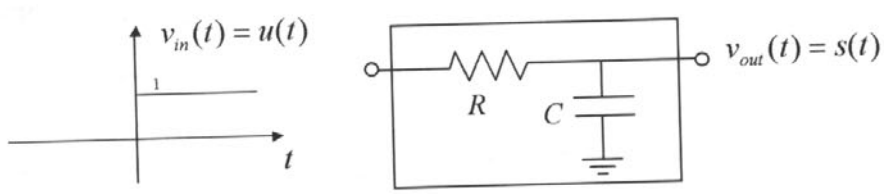

FIGURE 2.33 Setup for step response of an *RC* circuit.

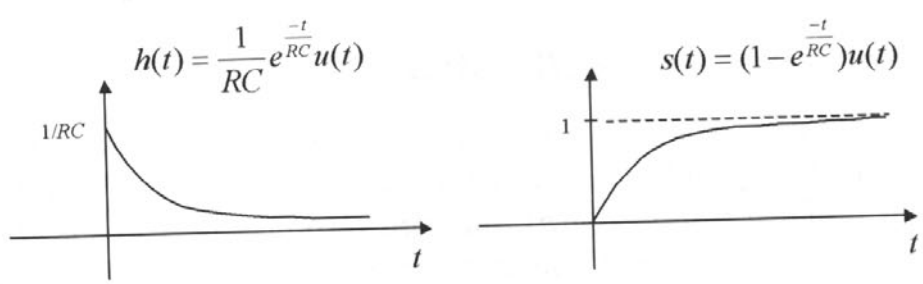

FIGURE 2.34 Impulse response and step response of the *RC* circuit.

With the assumption that $A \neq 0$, this equation holds if and only if the *characteristic polynomial* $p(s) := \sum_{k=0}^{N} a_k s^k$ vanishes at the complex number s:

$$p(s) = \sum_{k=0}^{N} a_k s^k = a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0 = 0. \tag{3.38}$$

By the fundamental theorem of algebra, Equation 3.38 has at most N distinct complex roots. Assume for simplicity that the N roots are distinct, and let them be denoted as $\{s_k\}_{k=1}^{N}$. This means that there are N distinct functions $A_k e^{s_k t}$ that satisfy the homogeneous Equation 3.36. Then, the solution to Equation 3.36 can be written as a linear combination of these complex exponentials:

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t}. \tag{3.39}$$

The N complex coefficients $\{A_k\}_{k=1}^{N}$ can be computed using the initial conditions:

$$\begin{aligned}
0 &= h_a(0^+) = \sum_{k=1}^{N} A_k e^{s_k 0^+} = \sum_{k=1}^{N} A_k \\
0 &= \frac{dh_a(0^+)}{dt} = \sum_{k=1}^{N} A_k s_k \\
&\vdots \\
\frac{1}{a_N} &= \frac{d^{N-1} h_a(0^+)}{dt^{N-1}} = \sum_{k=1}^{N} A_k s_k^{N-1}.
\end{aligned} \tag{3.40}$$

This set of linear equations can be written as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1/a_N \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_N \\ s_1^2 & s_2^2 & \cdots & s_N^2 \\ \vdots & \vdots & \vdots & \vdots \\ s_1^{N-1} & s_2^{N-1} & \cdots & s_N^{N-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_N \end{bmatrix}. \tag{3.41}$$

The $N \times N$ matrix with complex entries in this equation is called a *Vandermonde matrix* and it can be shown to be nonsingular (invertible). Thus, a unique solution always exists for the A_k's, which gives us the unique solution $h_a(t)$ through Equation 3.39.

Now, define the duty cycle $\eta := \frac{2t_0}{T}$ of the rectangular wave as the fraction of time the signal is "on" (equal to one) over one period. The duty cycle is often given as a percentage. The spectral coefficients expressed using the sinc function and the duty cycle can be written as

$$a_k = \frac{2t_0}{T} \frac{\sin\left(\frac{\pi k 2t_0}{T}\right)}{\frac{\pi k 2t_0}{T}} = \frac{2t_0}{T}\operatorname{sinc}\left(\frac{k2t_0}{T}\right)$$
$$= \eta \operatorname{sinc}(k\eta). \tag{4.21}$$

For a 50% duty cycle, that is, $\eta = \frac{1}{2}$, we get the Fourier series coefficients given in Equation 4.22 and whose line spectrum is shown in Figure 4.7. Note that the coefficients are real, so a single plot suffices. However, one could also choose to sketch the magnitude (absolute value of a_k) and the phase (0 for a_k nonnegative, $-\pi/2$ for a_k negative) on two separate graphs.

$-\pi$

$$a_k = \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right). \tag{4.22}$$

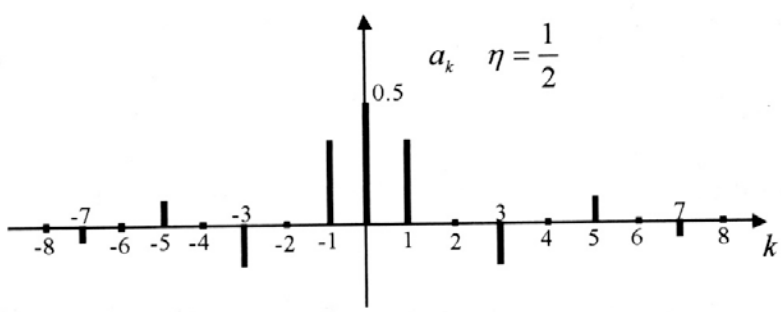

FIGURE 4.7 Spectral coefficients of the rectangular wave for a 50% duty cycle.

Remember that k is a multiple of the fundamental frequency. So for a 60 Hz (120π rad/s) square wave, the coefficients $a_{\pm 1}$ are the fundamental components at 60 Hz, $a_{\pm 2}$ are the second harmonic components at 120 Hz, etc.

For shorter duty cycles (shorter pulses with respect to the fundamental period), the "sinc envelope" of the spectral coefficients expands, and we get more coefficients in each lobe. For example, the real line spectrum of a rectangular wave with $\eta = \frac{1}{8}$ is shown in Figure 4.8.

Using Table D.1 of basic Fourier transform pairs in Appendix D, we find that

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t). \tag{5.34}$$

For the case where the two exponents are equal, for example, $h(t) = e^{-2t}u(t)$ and $x(t) = e^{-2t}u(t)$, the partial fraction expansion of Equation 5.33 is not valid. On the other hand, in this case the spectrum of $y(t)$ is given by $Y(j\omega) = \frac{1}{(j\omega+2)^2}$, which in the table corresponds to the signal $y(t) = te^{-2t}u(t)$. The partial fraction expansion technique will be reviewed in more detail in Chapter 6.

JALITY

The Fourier transform pair is quite symmetric. This results in a duality between the time domain and the frequency domain. For example, Figure 5.12 shows that a rectangular pulse signal in the time domain has a Fourier transform that takes the form of a sinc function of frequency. The dual of this situation is a rectangular spectrum that turns out to be the Fourier transform of a signal that is a sinc function of time.

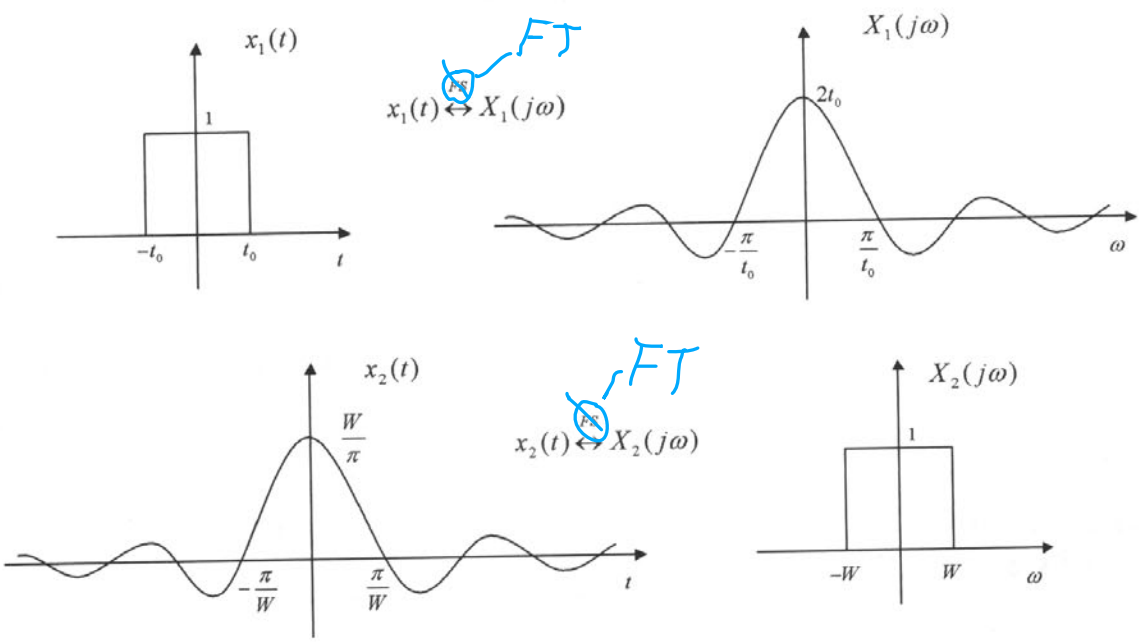

FIGURE 5.12 Duality between time-domain and frequency-domain functions.

When such a rectangular spectrum centered at $\omega = 0$ is the frequency response of an LTI system, it is often referred to as an *ideal lowpass filter* because it lets the low frequency components pass undistorted while the high frequencies are completely cut off. The problem is that the impulse response of this LTI system, the

Answer:

$$S_1: \quad (j\omega)Y(j\omega)+Y(j\omega)=X(j\omega)$$
$$\Rightarrow H_1(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{j\omega+1}$$
$$S_2: \quad (j\omega)Y(j\omega)+2Y(j\omega)=(j\omega)X(j\omega)+X(j\omega)$$
$$\Rightarrow H_2(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+1}{j\omega+2}$$

The overall closed-loop frequency response is obtained by first writing loop equations for the error signal $e(t)$ (output of the summing junction) and output.

$$E(j\omega)=X(j\omega)-H_2(j\omega)H_1(j\omega)E(j\omega)$$
$$Y(j\omega)=H_1(j\omega)E(j\omega)$$

Solving the first equation for $E(j\omega)$, we obtain

$$E(j\omega)=\frac{1}{1+H_2(j\omega)H_1(j\omega)}X(j\omega)$$
$$Y(j\omega)=\underbrace{\frac{H_1(j\omega)}{1+H_2(j\omega)H_1(j\omega)}}_{H(j\omega)}X(j\omega).$$

Thus,

$$H(j\omega)=\frac{H_1(j\omega)}{1+H_2(j\omega)H_1(j\omega)}=\frac{\dfrac{1}{j\omega+1}}{1+\dfrac{1}{j\omega+1}\dfrac{j\omega+1}{j\omega+2}}$$
$$=\frac{\dfrac{1}{j\omega+1}}{1+\dfrac{1}{j\omega+2}}=\frac{j\omega+2}{(j\omega+3)(j\omega+1)}.$$

Magnitude and phase:

Finally, coefficient A_3 is computed:

$$A_3 = (s-2)\left(\frac{(s+3)}{s(s+1)(s-2)}\right)_{s=2} = \frac{5}{(2)(3)} = \frac{5}{6}. \tag{6.20}$$

Hence, the Laplace transform can be expanded as

$$X(s) = \frac{2}{3}\underbrace{\frac{1}{s+1}}_{\mathrm{Re}\{s\}>-1} - \frac{3}{2}\underbrace{\frac{1}{s}}_{\mathrm{Re}\{s\}>0} + \frac{5}{6}\underbrace{\frac{1}{s-2}}_{\mathrm{Re}\{s\}<2}, \tag{6.21}$$

and from Table D.4 of Laplace transform pairs, we obtain the signal

$$x(t) = \frac{2}{3}e^{-t}u(t) - \frac{3}{2}u(t) - \frac{5}{6}e^{2t}u(-t). \tag{6.22}$$

For a multiple pole p_m of multiplicity r, the coefficients $A_m, \ldots, A_{m+r-1}$ are computed as follows:

$$A_{m+r-i} = \frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}\left[(s-p_m)^r X(s)\right]_{s=p_m}, i = 1, \ldots, r, \tag{6.23}$$

where $0! = 1$ by convention. To compute the coefficient of the term with the highest power of the repeated pole, we simply have to compute

$$A_{m+r-1} = \left[(s-p_m)^r X(s)\right]_{s=p_m}. \tag{6.24}$$

It should be clear that after multiplication by $(s-p_m)^r$ on both sides of Equation 6.14, all the terms on the right-hand side will vanish upon letting $s = p_m$, except the term $\frac{A_{m+r-1}}{(s-p_m)^r}$, which yields A_{m+r-1}. Now, consider the computation of A_{m+r-2} using the formula $A_{m+r-2} = \frac{d}{ds}\left[(s-p_m)^r X(s)\right]_{s=p_m}$. After multiplication by $(s-p_m)^r$, the terms on the right-hand side corresponding to the multiple pole become

$$(s-p_m)^{r-1}A_m + (s-p_m)^{r-2}A_{m+1} + \cdots + (s-p_m)A_{m+r-2} + A_{m+r-1}. \tag{6.25}$$

After differentiating with respect to s and letting $s = p_m$, we obtain

$$\left[(r-1)(s-p_m)^{r-2}A_m + (r-2)(s-p_m)^{r-3}A_{m+1} + \cdots + 2(s-p_m)A_{m+r-3} + A_{m+r-2}\right]_{s=p_m} = A_{m+r-2} \tag{6.26}$$

$$X(s)=\frac{2s^2+3s-2}{\left(s^2+2s+4\right)s}=\underbrace{\frac{A\sqrt{3}+B(s+1)}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}+\underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0}. \tag{6.38}$$

Coefficient C is easily obtained with the residue technique: $C=\frac{-1}{2}$. Now, let $s=-1$ to compute $\frac{-3}{-3}=\frac{1}{\sqrt{3}}A+\frac{1}{2}\Rightarrow A=\frac{\sqrt{3}}{2}$, and then multiply both sides by s and let $s\to\infty$ to get $B=5/2$. Then we have the following expansion:

$$\begin{aligned}X(s)&=\underbrace{\frac{\frac{\sqrt{3}}{2}(\sqrt{3})+\frac{5}{2}(s+1)}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}-\underbrace{\frac{1/2}{s}}_{\mathrm{Re}\{s\}>0}\\&=\underbrace{\frac{\frac{\sqrt{3}}{2}(\sqrt{3})}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}+\underbrace{\frac{\frac{5}{2}(s+1)}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}-\underbrace{\frac{1/2}{s}}_{\mathrm{Re}\{s\}>0}.\end{aligned} \tag{6.39}$$

Taking the inverse Laplace transform using Table D.4, we obtain

$$x(t)=\left[\frac{\sqrt{3}}{2}e^{-t}\sin(\sqrt{3}t)+\frac{5}{2}e^{-t}\cos(\sqrt{3}t)\right]u(t)-\frac{1}{2}u(t). \tag{6.40}$$

CONVERGENCE OF THE TWO-SIDED LAPLACE TRANSFORM

As mentioned above, the convergence of the integral in Equation 6.1 depends on the value of the real part of the complex Laplace variable. Thus, the ROC in the complex plane or s-plane is either a vertical half-plane, a vertical strip, or nothing. We have seen two examples above that led to open half-plane ROCs. Here is a signal for which the Laplace transform only converges in an open vertical strip.

Example 6.7: Consider the double-sided signal $x(t)=e^{-2t}u(t)+e^{t}u(-t)$ shown in Figure 6.5.

Differentiation in the Frequency Domain

If $x(t) \overset{UL}{\leftrightarrow} \mathcal{X}(s),\ s \in \text{ROC}$, then

$$-tx(t) \overset{UL}{\leftrightarrow} \frac{d\mathcal{X}(s)}{ds},\ s \in \text{ROC}. \tag{6.83}$$

Integration in the Time Domain

If $x(t) \overset{UL}{\leftrightarrow} \mathcal{X}(s),\ s \in \text{ROC}$, then

$$\int_0^t x(\tau)d\tau \overset{UL}{\leftrightarrow} \frac{1}{s}\mathcal{X}(s),\ s \in \text{ROC}_1 \supseteq \text{ROC} \cap \{s : \text{Re}\{s\} > 0\}. \tag{6.84}$$

The Initial and Final Value Theorems

Even though these theorems were introduced as two-sided Laplace transform properties, they are basically unilateral transform properties, as they apply only to signals that are identically 0 for $t < 0$.

The *initial-value theorem* states that

$$x(0^+) = \lim_{s \to +\infty} s\mathcal{X}(s), \tag{6.85}$$

and the *final-value theorem* states that

$$\lim_{\cancel{s \to +\infty}} x(t) = \lim_{s \to 0} s\mathcal{X}(s). \tag{6.86}$$

$t \to +\infty$

Example 6.16: Let us find the initial value $x(0^+)$ of the signal whose unilateral Laplace transform is $\mathcal{X}(s) = \frac{10}{s-3}$, $\text{Re}\{s\} > 3$:

$$x(0^+) = \lim_{s \to +\infty} s\mathcal{X}(s) = \lim_{s \to +\infty} s\frac{10}{s-3} = 10. \tag{6.87}$$

SUMMARY

In this chapter, we introduced the Laplace transform of a continuous-time signal as a generalization of the Fourier transform.

The break frequencies are at 1, 10, and 100 rad/s. For the Bode magnitude plot, we can sketch the asymptotes of each first-order term in Equation 8.19 on the same magnitude graph (as dashed lines) and then add them together as shown in Figure 8.9.

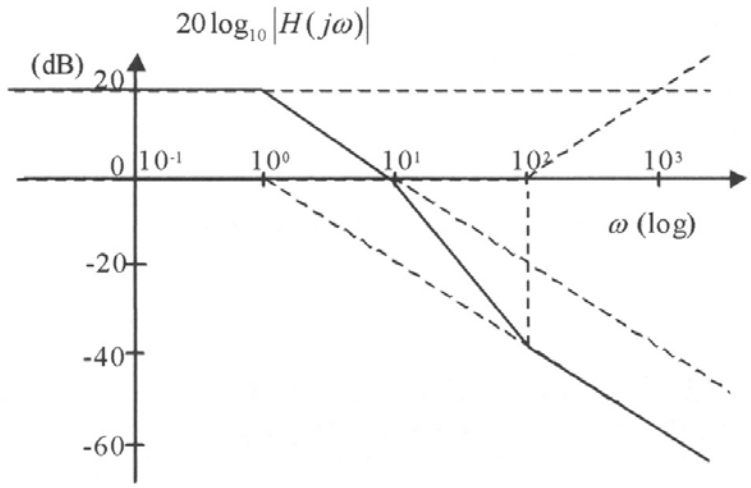

FIGURE 8.9 Bode magnitude plot of a second-order example.

We proceed in a similar fashion to obtain the Bode phase plot of Figure 8.10.

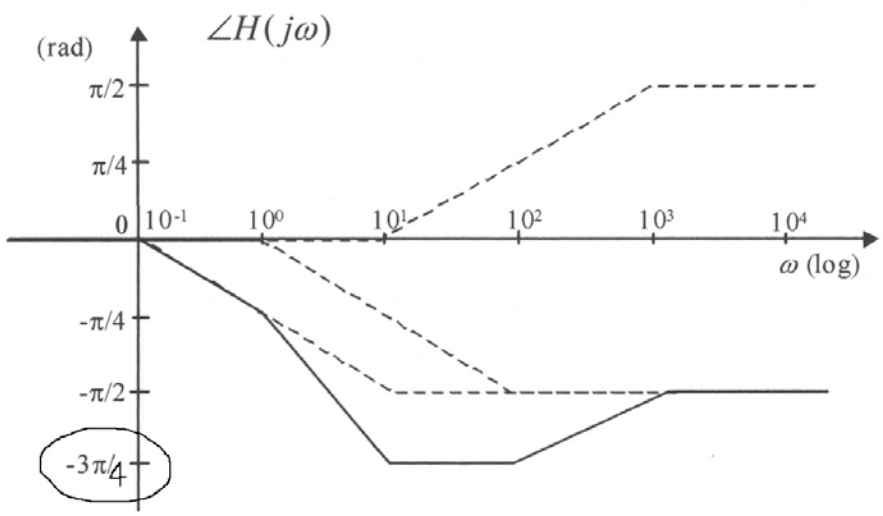

FIGURE 8.10 Bode phase plot of a second-order example.

A Bode plot can easily be obtained in MATLAB. The following MATLAB script bodeplot.m, which is located on the CD-ROM in D:\Chapter8, produces the Bode plot of Example 8.4.

ON THE CD

Given the source current $i_s(t)$, the voltage $v(t) = v_2(t)$ is obtained by solving the first-order differential Equation (9.2). Suppose there is an initial condition $v(0^-)$ on the capacitor voltage and the current is a unit step function. We use the unilateral Laplace transform to obtain

$$Cs\mathcal{V}(s) - Cv(0^-) + \frac{1}{R}\mathcal{V}(s) = \cancel{\mathcal{V}_s(s)} \quad I_s(s)$$

$$\Rightarrow \quad \mathcal{V}(s) = \frac{R}{RCs+1}\cancel{\mathcal{V}_s(s)} + \frac{RC}{RCs+1}v(0^-)$$

$$= \frac{R}{s(RCs+1)} + \frac{RC}{RCs+1}v(0^-)$$

$$= \frac{R}{s} + \frac{-R}{s+\frac{1}{RC}} + \frac{1}{s+\frac{1}{RC}}v(0^-). \tag{9.3}$$

Taking the inverse Laplace transform, we get

$$v(t) = R(1 - e^{-\frac{t}{RC}})u(t) + v(0^-)e^{-\frac{t}{RC}}u(t) \tag{9.4}$$

When voltage sources are present in the circuit, we can create "supernodes" around them. KCL applies to these supernodes. However, there are two node voltages associated with them, not just one.

Example 9.2: Consider the *RLC* circuit of Figure 9.2.

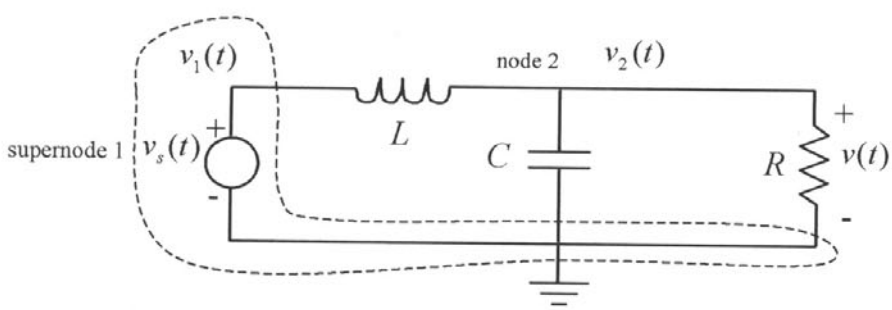

FIGURE 9.2 *RLC* circuit with voltage source input and supernode.

Since there is one supernode for which we know that $v_1(t) = v_s(t)$ and one ordinary node, we need only one node equation:

$$(\lambda_1 I - A)v_1 = \begin{bmatrix} -1+j+1 & -1 \\ 1 & -1+j+1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} j & -1 \\ 1 & j \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_{11} = 1, v_{12} = j$$

$$v_2 = v_1^* = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

Thus, $T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$, $T^{-1} = \frac{1}{-2j}\begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -j\frac{1}{2} \\ \frac{1}{2} & j\frac{1}{2} \end{bmatrix}$, and

$$h(t) = CT\text{diag}\{e^{(-1+j)t}, e^{(-1-j)t}\}T^{-1}\overset{B}{\cancel{q}}(t)$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} e^{(-1+j)t} & 0 \\ 0 & e^{(-1-j)t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -j\frac{1}{2} \\ \frac{1}{2} & j\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}q(t)$$

$$= \begin{bmatrix} 1+j & 1-j \end{bmatrix}\begin{bmatrix} e^{(-1+j)t} & 0 \\ 0 & e^{(-1-j)t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}q(t)$$

$$= \left[\left(\frac{1}{2} + j\frac{1}{2}\right)e^{(-1+j)t} + \left(\frac{1}{2} - j\frac{1}{2}\right)e^{(-1-j)t}\right]q(t)$$

$$= e^{-t}\left[\left(\frac{1}{2} + j\frac{1}{2}\right)e^{jt} + \left(\frac{1}{2} - j\frac{1}{2}\right)e^{-jt}\right]q(t)$$

$$= 2e^{-t}\,\text{Re}\left[\left(\frac{1}{2} + j\frac{1}{2}\right)e^{jt}\right]q(t) = e^{-t}(\cos t - \sin t)q(t),$$

Thus, $h(t) = e^{-t}(\cos t - \sin t)q(t)$.

Exercise 10.2

Find the controllable and observable canonical state-space realizations for the following LTI system:

$$H(s) = \frac{s^3 + s + 2}{s^3 + 3s}, \quad \text{Re}\{s\} > 0.$$

reference must have most of its energy in the passband of the transmission. It is indeed the case, as seen in the plot of $|\hat{y}_d(j\omega)|$ in Figure 11.14:

$$\hat{y}_d(j\omega) = \frac{10e^{-j5\omega}}{\pi}\text{sinc}\left(\frac{5\omega}{\pi}\right). \tag{11.24}$$

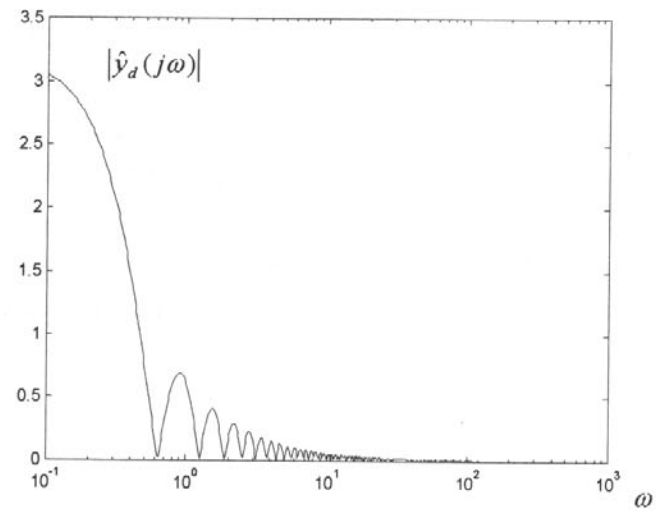

FIGURE 11.14 Magnitude of the input pulse's spectrum.

A Naive Approach to Controller Design

Given a plant model $P(s)$ and a desired stable closed-loop sensitivity or transmission, it is often possible to back solve for the controller $K(s)$ using Equation 11.9 or 11.19.

Example 11.4: A plant is modeled by the transfer function $P(s) = \frac{2}{s+1}$. Let us find the controller that will yield the desired complementary sensitivity function $T(s) = \frac{10}{s+10}$. From Equation 11.19, we have

$$K(s) = \frac{T(s)}{P(s)[1-T(s)]} = \frac{\frac{10}{s+10}}{\frac{2}{s+1}\left[1-\frac{10}{s+10}\right]} = \frac{5(s+1)}{s(s+10)}. \tag{11.25}$$

Thus, we can write the DTFT of a periodic signal by inspection from the knowledge of its Fourier series coefficients (recall that they are periodic).

Example 12.6: Let us find the DTFT of $x[n]=1+\sin(\frac{2\pi}{3}n)-\cos(\frac{2\pi}{7}n)$. We first have to determine whether this signal is periodic. The sine term repeats every three time steps, whereas the cosine term repeats every seven time steps. Thus, the signal is periodic of fundamental period $N=21$. Write

$$x[n]=1+\frac{1}{2j}\left(e^{j\frac{2\pi 7}{21}n}-e^{-j\frac{2\pi 7}{21}n}\right)-\frac{1}{2}\left(e^{j\frac{2\pi 3}{21}n}+e^{-j\frac{2\pi 3}{21}n}\right). \tag{12.68}$$

The nonzero DTFS coefficients of the signal are $a_0=1, a_{-7}=j\frac{1}{2}, a_7=-j\frac{1}{2}$, $a_{-3}=a_3=\frac{-1}{2}$; hence we have

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{k=-10}^{10} 2\pi a_k \sum_{l=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{21}-2\pi l)\\
&= 1-\pi\sum_{l=-\infty}^{\infty}\delta(\omega+\frac{6\pi}{21}-2\pi l)-\pi\sum_{l=-\infty}^{\infty}\delta(\omega-\frac{6\pi}{21}-2\pi l)\\
&\quad +j\pi\sum_{l=-\infty}^{\infty}\delta(\omega+\frac{14\pi}{21}-2\pi l)-j\pi\sum_{l=-\infty}^{\infty}\delta(\omega-\frac{14\pi}{21}-2\pi l)\\
&= 1+\sum_{l=-\infty}^{\infty}\left[-\pi\delta(\omega+\frac{6\pi}{21}-2\pi l)-\pi\delta(\omega-\frac{6\pi}{21}-2\pi l)+j\pi\delta(\omega+\frac{14\pi}{21}-2\pi l)-j\pi\delta(\omega-\frac{14\pi}{21}-2\pi l)\right].
\end{aligned} \tag{12.69}$$

DTFT of a Periodic Discrete-Time Impulse Train

Let us now find the DTFT of the discrete-time impulse train $x[n]=\sum_{k=-\infty}^{\infty}\delta[n-kN]$. This signal is periodic of period N. Its DTFS coefficients are given by

$$a_k=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\frac{2\pi}{N}n}=\frac{1}{N}\sum_{n=0}^{N-1}\delta[n]e^{-jk\frac{2\pi}{N}n}=\frac{1}{N}, \tag{12.70}$$

and hence, the DTFT of the impulse train is simply a train of impulses in the frequency domain, equally spaced by $\frac{2\pi}{N}$ radians.

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{k=0}^{N-1}\frac{2\pi}{N}\sum_{l=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{N}-2\pi l)\\
&= \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{N})
\end{aligned} \tag{12.71}$$

This transform is shown in Figure 12.12.

Its impulse response is

$$h[n] = a^n u[n], \tag{14.14}$$

and its unit step response is computed as the running sum of its impulse response:

$$s[n] = \sum_{k=-\infty}^{n} a^k u[k] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} u[n]. \tag{14.15}$$

The magnitude of the parameter a plays a role similar to that of the time constant τ of a continuous-time first-order system. Specifically, $|a|$ determines the rate at which the system responds. For small $|a|$, the impulse response decays sharply and the step response settles quickly. For $|a|$ close to 1, the transients are much slower. Figure 14.10 shows the step responses of two first-order systems, one with $a = 0.5$ and the other with $a = 0.8$.

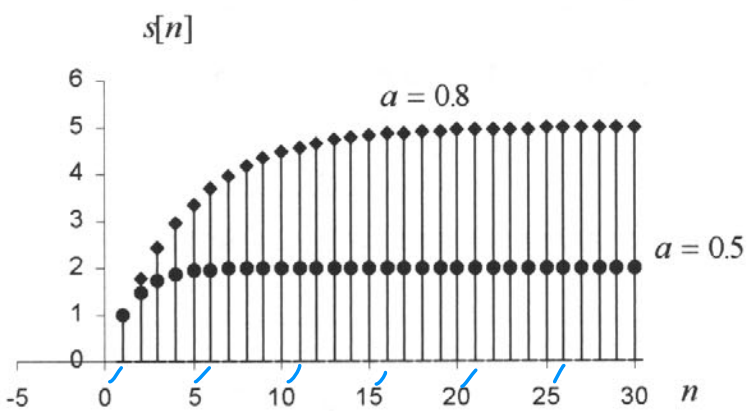

FIGURE 14.10 Step responses of two first-order DLTI systems with a real positive pole.

From a frequency point of view, the lowpass frequency response of the system corresponding to small $|a|$ has a wider bandwidth than for $|a|$ close to 1. This is easier to see when the frequency response $H(e^{j\omega})$ is normalized to have a unity DC gain:

$$\begin{aligned} |H(e^{j\omega})| &= \left|(1-a)\frac{1}{1-ae^{-j\omega}}\right| \\ &= \frac{1-a}{(1-a\cos\omega)^2 + (a\sin\omega)^2} \\ &= \frac{1-a}{(1+a^2-2a\cos\omega)^{1/2}} \end{aligned} \tag{14.16}$$

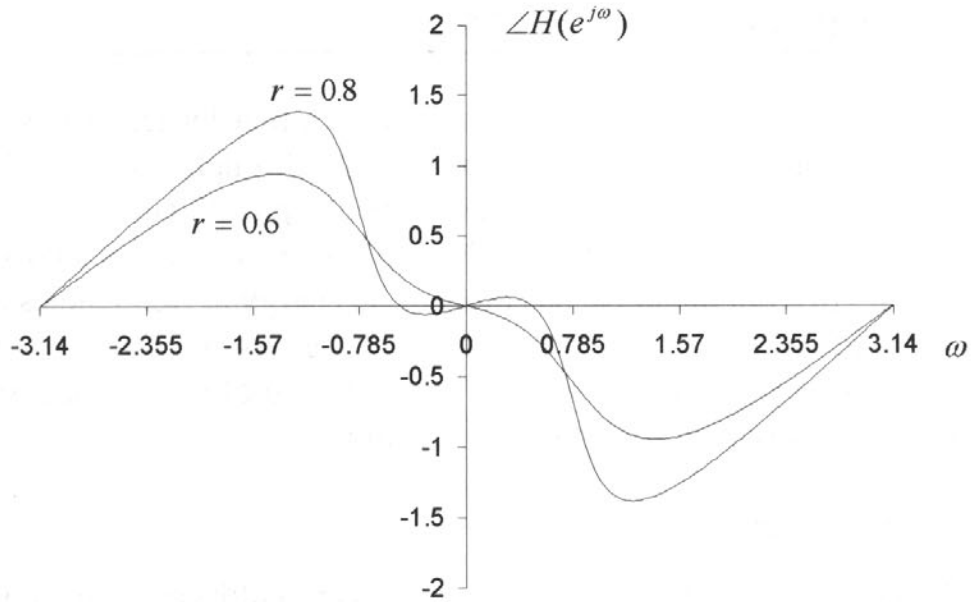

FIGURE 14.14 Frequency response phase of second-order transfer functions with two pole magnitudes.

This impulse response is shown in Figure 14.15, again for the cases $r = 0.6$ and $r = 0.8$.

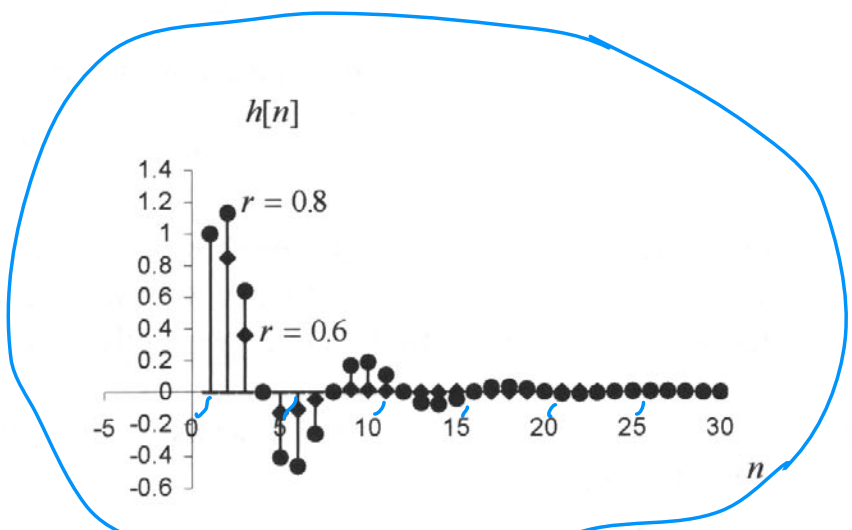

FIGURE 14.15 Impulse responses of second-order transfer functions with two pole magnitudes.

We can see that the impulse response for $r = 0.8$ displays larger oscillations, which corresponds to the higher peaks in the frequency response.

((Lecture 55: Ideal Discrete-Time Filters))

Second-Order Approximation

The second-order frequency response is normalized at the frequencies $\pm\omega_p$ wher the peaks occur. These frequencies are found to be:

$$\omega_p = \pm\arccos\left\{\frac{1+r^2}{2r}\cos\theta\right\}. \tag{14.39}$$

Thus, the normalized magnitude of the second-order frequency response is

$$\left|H(e^{j\omega})\right| = \frac{\left[1+r^2-2r\cos(\omega_p+\theta)\right]^{\frac{1}{2}}\left[1+r^2-2r\cos(\omega_p-\theta)\right]^{\frac{1}{2}}}{\left[1+r^2-2r\cos(\omega+\theta)\right]^{\frac{1}{2}}\left[1+r^2-2r\cos(\omega-\theta)\right]^{\frac{1}{2}}}. \tag{14.40}$$

We can use this expression to carry out a design by trial and error. Suppose w are given the frequencies ω_{c1}, ω_{c2} of the ideal bandpass filter that we have to ap proximate with a second-order filter. We first set $\omega_p = \frac{\omega_{c1}+\omega_{c2}}{2}$, and we compute th pole angle θ by using the inverse of the relationship in Equation 14.39:

$$\cos\theta = \frac{2r}{1+r^2}\cos\omega_p$$

$$\Rightarrow$$

$$\theta = \arccos\left\{\frac{2r}{1+r^2}\cos\omega_p\right\}. \tag{14.41}$$

Then, we can try out different values of $0 < r < 1$ to obtain the proper passband

Example 14.4: Let us design a second-order bandpass filter approximating th ideal bandpass frequency response with $\omega_{c1} = \frac{\pi}{4}$, $\omega_{c1} = \frac{\pi}{2}$. We compute $\omega_p = \frac{\omega_{c1}+\omega_{c2}}{2} =$ $\frac{3\pi}{8} = 1.1781$. Table 14.2 shows the pole angle for different values of r.

c2

TABLE 14.2 Magnitude and Angle of Complex Pole for the Design of a Second-Order Bandpass Filter.

r	0.6	0.7	0.8	0.9
θ	1.226366	1.202992	1.18818	1.180386

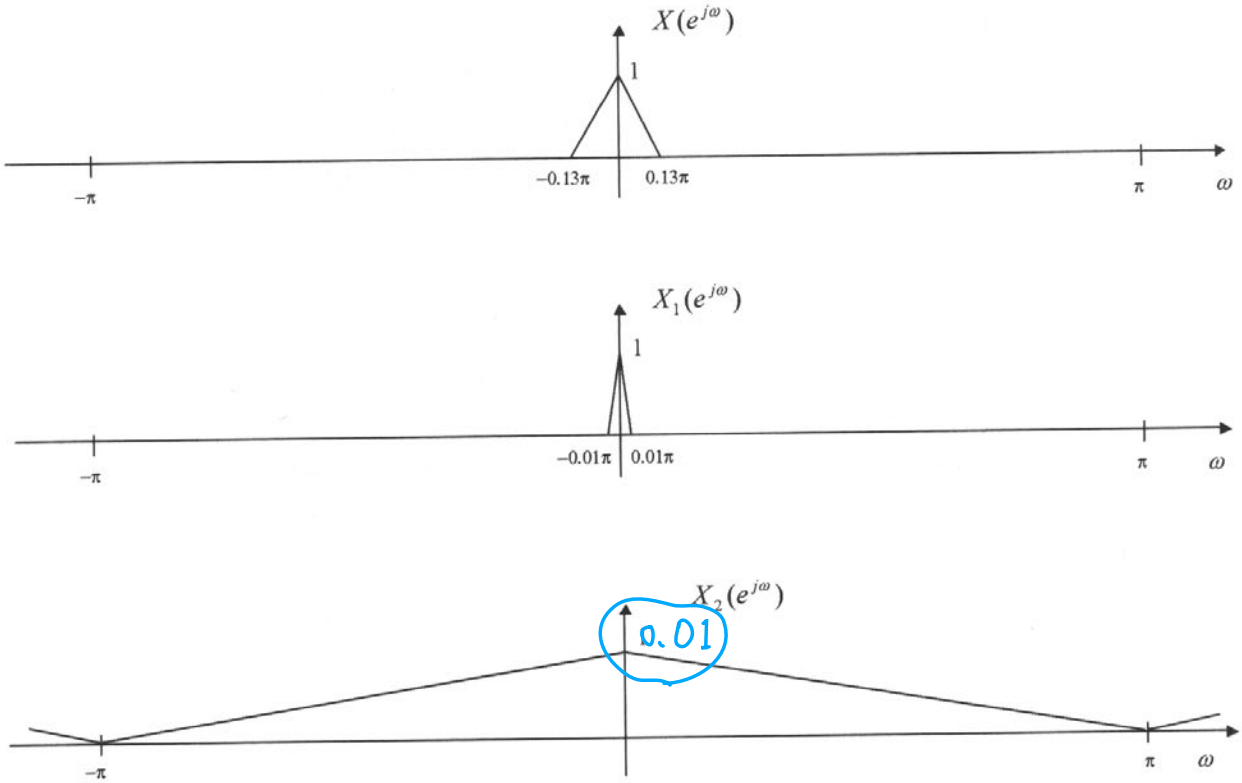

FIGURE 15.36 DTFTs of upsampled filtered and decimated signals (Exercise 15.2).

Exercises

Exercise 15.3

The signals below are sampled with sampling period T_s. Determine the bounds on T_s that guarantee there will be no aliasing.

(a) $x(t) = \cos(10\pi t)\dfrac{\sin(\pi t)}{2t}$

(b) $x(t) = e^{-4t}u(t) * \dfrac{\sin(Wt)}{\pi t}$

Answer:

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The DC gain of this closed-loop transfer function is 1, which represents near-perfect tracking of the phase of the FM signal when it changes slowly. The magnitude of the Bode plot of $T(s)$ stays close to 1 (0 dB) up until the break frequency $\omega_0 = kk_{VCO}$. This frequency must be made higher than the bandwidth of the reference signal, which is essentially the bandwidth of the message signal (modulo the $1/j\omega$ effect of integration), by selecting the controller gain k appropriately. Also recall that the linear approximation of the sine function of the phase error holds for small angles only. This is another important consideration to keep in mind in the design of $K(s)$, as the PLL could "unlock" if the error becomes too large.

Finally, the message-bearing signal of interest in the PLL is the output of the controller, which is given by

$$\hat{\tilde{x}}(s) = \frac{k\,\hat{r}(s)}{1 + k\dfrac{k_{VCO}}{s}} = \frac{ks}{s + kk_{VCO}}\hat{r}(s) = \frac{1}{k_{VCO}}\frac{s}{\dfrac{1}{kk_{VCO}}s + 1}\hat{r}(s). \tag{16.57}$$

Thus, signal $\tilde{x}(t)$ is a filtered version of $\frac{d}{dt} \cdot r(t) = \frac{k_f}{k_{VCO}}\, x(t)$, where the derivative comes from the numerator s, but the effect of the first-order lowpass filter $\frac{1}{\frac{1}{kk_{VCO}}s+1}$ can be neglected if the controller gain k is chosen high enough, which pushes the cutoff frequency outside of the bandwidth of the message signal. Therefore, with a large controller gain, the first-order PLL effectively demodulates the FM signal by giving the voltage signal:

$$\tilde{x}(t) \cong \frac{k_f}{k_{VCO}} x(t). \tag{16.58}$$

SUMMARY

This chapter was a brief introduction to the theory of communication systems.

- Amplitude modulation of a message signal is obtained by multiplying the signal with a sinusoidal carrier. This shifts the spectrum of the message and recenters it around the carrier frequency. The AM signal can be demodulated by a synchronous demodulator whose frequency must be made equal to the carrier frequency, or by a simpler asynchronous demodulator essentially consisting of an envelope detector.

and the spectrum of the modulated signal is found to be

$$Y(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{0.4}{(1-4k^2)}\delta(\omega - 200000\pi - k2000\pi) + \sum_{k=-\infty}^{+\infty} \frac{0.4}{(1-4k^2)}\delta(\omega + 200000\pi - k2000\pi)$$
$$+\frac{1}{2}\delta(\omega+200000\pi) + \frac{1}{2}\delta(\omega-200000\pi)$$

This spectrum is sketched in Figure 16.29.

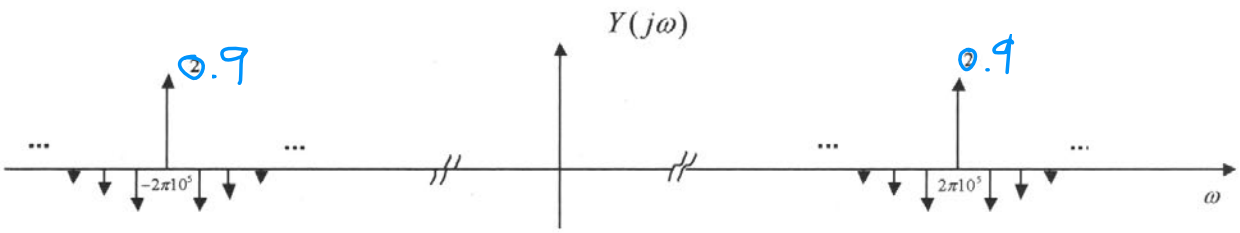

FIGURE 16.29 Spectrum of modulated signal of Exercise 16.1.

(b) Design an envelope detector to demodulate the AM signal. That is, draw a circuit diagram of the envelope detector and compute the values of the circuit components. Justify all of your approximations and assumptions. Provide rough sketches of the carrier signal, the modulated signal, and the signal at the output of the detector. What is the modulation index m of the AM signal?

Answer:
An envelope detector can be implemented with the simple RC circuit with a diode shown in Figure 16.30.

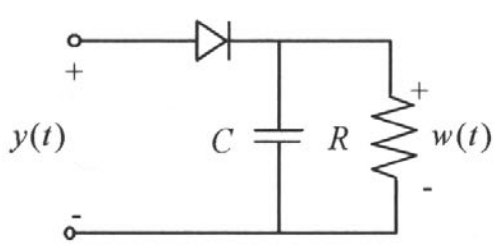

FIGURE 16.30 Envelope detector in Exercise 16.1.

zero-input response due to the initial conditions only. For a discrete-time state-space system, the initial conditions are captured in the *initial state* $x[0]$.

Zero-Input Response

The zero-input response of a state-space system is the response to a nonzero initial state only. Consider the following general state-space system:

$$\begin{aligned} x[n+1] &= Ax[n] + Bu[n], \\ y[n] &= Cx[n] + Du[n] \end{aligned} \tag{17.17}$$

where (the values, not the signals) $x[n] \in \mathbb{R}^N$, $y[n] \in \mathbb{R}$, $u[n] \in \mathbb{R}$ and $A \in \mathbb{R}^{N \times N}, B \in \mathbb{R}^{N \times 1}, C \in \mathbb{R}^{1 \times N}, D \in \mathbb{R}$, with initial state $x[0] \neq 0$ and input $u[n] = 0$. Let the unit step signal be denoted as $q[n]$. A solution can be readily obtained recursively:

$$\begin{aligned} x[1] &= Ax[0] \\ y_{zi}[0] &= Cx[0] \\ x[2] &= A^2 x[0] \\ y_{zi}[1] &= CAx[0] \\ x[3] &= A^3 x[0] \\ y_{zi}[2] &= CA^2 x[0] \\ &\vdots \\ x[n+1] &= A^{n+1} x[0] \\ y_{zi}[n] &= CA^n x[0]. \end{aligned} \tag{17.18}$$

Hence, the zero-input response is $y_{zi}[n] = CA^n x[0] q[n]$, and the corresponding state response is $x[n] = A^n x[0] q[n]$.

Zero-State Response

The zero-state response $y_{zs}[n]$ is the response of the system to the input only (zero initial conditions). A recursive solution yields

Appendix D Tables of Transforms

TABLE D.1 Fourier Transform Pairs

Time domain $x(t)$		**Frequency domain** $X(j\omega)$
$\delta(t)$		1
$u(t)$		$\frac{1}{j\omega}+\pi\delta(\omega)$
$e^{at}u(t)$	$a\in\mathbb{C}, \mathrm{Re}\{a\}<0$	$\frac{1}{j\omega-a}$
$-e^{at}u(-t)$	$a\in\mathbb{C}, \mathrm{Re}\{a\}>0$	$\frac{1}{j\omega-a}$
$te^{at}u(t)$	$a\in\mathbb{C}, \mathrm{Re}\{a\}<0$	$\frac{1}{(j\omega-a)^2}$
$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\alpha,\omega_0\in\mathbb{R},\ \alpha>0$	$\frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2}$
$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\alpha,\omega_0\in\mathbb{R},\ \alpha>0$	$\frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2}$
1		$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$\omega_0\in\mathbb{R}$	$2\pi\delta(\omega-\omega_0)$
$\sum_{-\infty}^{+\infty}\delta(t-kT)$	$T\in\mathbb{R},\ T>0$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\frac{2\pi}{T})$
$\begin{cases}1, & \lvert t\rvert<t_0\\ 0, & \lvert t\rvert>t_0\end{cases}$	$t_0\in\mathbb{R},\ t_0>0$	$2t_0\mathrm{sinc}(\frac{t_0}{\pi}\omega)$
$\frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c}{\pi}t\right)=\frac{\sin(\omega_c t)}{\pi t}$	$\omega_c\in\mathbb{R},\ \omega_c>0$	$\begin{cases}1, & \lvert\omega\rvert<\omega_c\\ 0, & \lvert\omega\rvert>\omega_c\end{cases}$
$\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}$ (Fourier series)	$a_k\in\mathbb{C}$, $\omega_0\in\mathbb{R},\ \omega_0>0$	$\sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-k\omega_0)$

TABLE D.4 Laplace Transform Pairs

Time domain $x(t)$		**Laplace domain** $X(s)$	ROC
$x(t)=\frac{1}{j2\pi}\int_{\alpha-j\infty}^{\alpha+j\infty} X(s)e^{st}ds$	$\{s\in\mathbb{C}:\mathrm{Re}\{s\}=\alpha\}$ $\in ROC$	$X(s)=\int_{-\infty}^{+\infty} x(t)e^{-st}dt$	$s\in\mathrm{ROC}$
$\delta(t)$		1	$\forall s$
$u(t)$		$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
$tu(t)$		$\frac{1}{s^2}$	$\mathrm{Re}\{s\}>0$
$t^k u(t)$	$k=1,2,3,\ldots$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}\{s\}>0$
$e^{at}u(t)$	$a\in\mathbb{C}$	$\frac{1}{s-a}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{a\}$
$-e^{at}u(-t)$	$a\in\mathbb{C}$	$\frac{1}{s-a}$	$\mathrm{Re}\{s\}<\mathrm{Re}\{a\}$
$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\alpha,\omega_0\in\mathbb{R}$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-\alpha$
$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\alpha,\omega_0\in\mathbb{R}$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-\alpha$
$\sin(\omega_0 t)u(t)$	$\omega_0\in\mathbb{R}$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\cos(\omega_0 t)u(t)$	$\omega_0\in\mathbb{R}$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\begin{cases}1, & \lvert t\rvert<t_0\\ 0, & \lvert t\rvert>t_0\end{cases}$	$t_0\in\mathbb{R},\ t_0>0$	$\frac{e^{t_0 s}-e^{-t_0 s}}{s}$	$x(t)$
$te^{at}u(t)$	$a\in\mathbb{C}$	$\frac{1}{(s-a)^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{a\}$

TABLE D.5: Properties of the Laplace Transform

Time domain		Laplace domain	ROC
$x(t),\ y(t)$		$X(s),\ Y(s)$	$\mathrm{ROC}_X, \mathrm{ROC}_Y$
$ax(t)+by(t)$	$a,b \in \mathbb{C}$	$aX(s)+bY(s)$	$\mathrm{ROC} \supseteq \mathrm{ROC}_X \cap \mathrm{ROC}_Y$
$x(t-t_0)$	$t_0 \in \mathbb{R}$	$e^{-st_0}X(s)$	ROC_X
$x(\alpha t)$	$\alpha \in \mathbb{R}$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$	$\frac{1}{\alpha}\mathrm{ROC}_X$
$\frac{d}{dt}x(t)$		$sX(s)$	$\mathrm{ROC} \supseteq \mathrm{ROC}_X$
$\int_{-\infty}^{t} x(\tau)d\tau$		$\frac{1}{s}X(s)$	$\mathrm{ROC} \supseteq \mathrm{ROC}_X \cap \{s : \mathrm{Re}\{s\} > 0\}$
$e^{s_0 t}x(t)$	$s_0 \in \mathbb{C}$	$X(s-s_0)$	$\mathrm{ROC}_X + \mathrm{Re}\{s_0\}$
$x(t) * y(t)$		$X(s)Y(s)$	$\mathrm{ROC} \supseteq \mathrm{ROC}_X \cap \mathrm{ROC}_Y$
$x^*(t)$		$X^*(s^*)$	ROC_X
$x(0^+)$	$x(t)=0, t<0$	$x(0^+) = \lim_{s\to+\infty} sX(s)$	Initial value theorem
$\lim_{t\to+\infty} x(t)$	$x(t)=0, t<0,$ $\left\lvert\lim_{t\to+\infty} x(t)\right\rvert < \infty$	$\lim_{t\to+\infty} x(t) = \lim_{s\to 0} sX(s)$	Final value theorem
$-tx(t)$		$\frac{dX(s)}{ds}$	ROC_X

TABLE D.6 Properties of the Unilateral Laplace Transform

Time domain $x(t)$		**Laplace domain** $X(s)$	ROC
$x(t),\ y(t),$ $x(t)=y(t)=0, t<0$		$\mathcal{X}(s),\ \mathcal{Y}(s)$	$\mathrm{ROC}_X, \mathrm{ROC}_Y$
$ax(t)+by(t)$	$a,b\in\mathbb{C}$	$a\mathcal{X}(s)+b\mathcal{Y}(s)$	$\mathrm{ROC}\supseteq\mathrm{ROC}_X\cap\mathrm{ROC}_Y$
$x(t-t_0)$	$t_0\in\mathbb{R}, t_0>0$	$e^{-st_0}\mathcal{X}(s)$	ROC_X
$x(\alpha t)$	$\alpha\in\mathbb{R}, \alpha>0$	$\frac{1}{\alpha}\mathcal{X}\left(\frac{s}{\alpha}\right)$	$\frac{1}{\alpha}\mathrm{ROC}_X$
$\frac{d}{dt}x(t)$		$s\mathcal{X}(s)-x(0^-)$	$\mathrm{ROC}\supseteq\mathrm{ROC}_X$
$\int_0^t x(\tau)d\tau$		$\frac{1}{s}\mathcal{X}(s)$	$\mathrm{ROC}\supseteq\mathrm{ROC}_X\cap\{s:\mathrm{Re}\{s\}>0\}$
$e^{s_0t}x(t)$	$s_0\in\mathbb{C}$	$\mathcal{X}(s-s_0)$	$\mathrm{ROC}_X+\mathrm{Re}\{s_0\}$
$x(t)*y(t)$		$\mathcal{X}(s)\mathcal{Y}(s)$	$\mathrm{ROC}\supseteq\mathrm{ROC}_X\cap\mathrm{ROC}_Y$
$x^*(t)$		$\mathcal{X}^*(s^*)$	ROC_X
$x(0^+)$		$x(0^+)=\lim_{s\to+\infty}s\mathcal{X}(s)$	Initial value theorem
$\lim_{t\to+\infty}x(t)$	$\left\lvert\lim_{t\to+\infty}x(t)\right\rvert<\infty$	$\lim_{t\to+\infty}x(t)=\lim_{s\to0}s\mathcal{X}(s)$	Final value theorem
$-tx(t)$		$\frac{d\mathcal{X}(s)}{ds}$	ROC_X

TABLE D.7 Discrete-Time Fourier Transform Pairs

Time domain $x[n]$		**Frequency domain** $X(e^{j\omega})$ always periodic of period 2π
$x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$		$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$
$\delta[n]$		1
$\delta[n-n_0]$	$n_0 \in \mathbb{Z}$	$e^{-j\omega n_0}$
$u[n]$		$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$
1		$\sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega-2\pi k)$
$u[n+n_0]-u[n-n_0-1]$	$n_0 \in \mathbb{Z}, n_0>0$	$\frac{\sin\omega(n_0+1/2)}{\sin(\omega/2)}$
$a^n u[n]$	$a \in \mathbb{C},\ \lvert a\rvert<1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u[n]$	$a \in \mathbb{C},\ \lvert a\rvert<1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$-a^n u[-n-1]$	$a \in \mathbb{C},\ \lvert a\rvert>1$	$\frac{1}{1-ae^{-j\omega}}$
$r^n\cos(\omega_0 n)u[n]$	$r,\omega_0 \in \mathbb{R}, r>0$	$\frac{1-r\cos(\omega_0)e^{-j\omega}}{1-2r\cos(\omega_0)e^{-j\omega}+r^2e^{-j2\omega}}$
$r^n\sin(\omega_0 n)u[n]$	$r,\omega_0 \in \mathbb{R}, r>0$	$\frac{r\sin(\omega_0)e^{-j\omega}}{1-2r\cos(\omega_0)e^{-j\omega}+r^2e^{-j2\omega}}$
$\cos(\omega_0 n)$	$\omega_0 \in \mathbb{R}$	$\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)$
$\sin(\omega_0 n)$	$\omega_0 \in \mathbb{R}$	$j\pi\sum_{k=-\infty}^{+\infty}-\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)$
$\frac{\sin\omega_c n}{\pi n}=\frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right)$	$\omega_c \in \mathbb{R}$, $0<\omega_c<\pi$	$\begin{cases}1, & \lvert\omega-k2\pi\rvert\le\omega_c, k\in\mathbb{Z}\\ 0, & \text{otherwise}\end{cases}$

→

TABLE D.10 *z*-Transform Pairs

Time domain $x[n]$		**z domain** $X(z)$	ROC
$x[n]=\frac{1}{j2\pi}\oint_{C} X(z)z^{n-1}dz$	$C := \{z \in \text{ROC} : \lvert z\rvert = r\}$	$X(z)=\sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	$z \in \text{ROC}$
$\delta[n]$		1	$\forall z$
$\delta[n-n_0]$	$n_0 \in \mathbb{Z}$	z^{-n_0}	$\forall z / \{0\},\ n_0 > 0$ $\forall z / \{\infty\},\ n_0 < 0$
$u[n]$		$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$u[n+n_0]-u[n-n_0-1]$	$n_0 \in \mathbb{Z}$	$\frac{z^{n_0}-z^{-n_0-1}}{1-z^{-1}}$	$\forall z / \{0,\infty\}$
$a^n u[n]$	$a \in \mathbb{C},\ \lvert a\rvert < 1$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$a \in \mathbb{C},\ \lvert a\rvert > 1$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$a \in \mathbb{C},\ \lvert a\rvert < 1$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$r^n \cos(\omega_0 n)u[n]$	$r,\omega_0 \in \mathbb{R},\ r > 0$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$r^n \sin(\omega_0 n)u[n]$	$r,\omega_0 \in \mathbb{R},\ r > 0$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$