Lecture 5: Transformations

2.5 Transformations

2.5.1 Definition: Transformation

A *transformation* is a function $\mathcal{A}:\mathcal{U}\to\mathcal{V}$ whose domain $\mathcal{D}\{\mathcal{A}\}$ and codomain $\mathcal{C}\{\mathcal{A}\}$ lie in vector spaces over the same field.

Such as transformation is called *linear* if for all $u, u_1, u_2 \in \mathcal{D}\{\mathcal{A}\}$ and $\alpha \in \mathcal{F}$,

$$\begin{aligned}\mathcal{A}(u_1+u_2)&=\mathcal{A}(u_1)+\mathcal{A}(u_2)\\ \mathcal{A}(\alpha u)&=\alpha\mathcal{A}(u)\end{aligned}$$

Examples:

(a) $\mathcal{A}:\mathbb{R}^n\to\mathbb{R}^m,\quad Au=v\quad \text{(matrix multiplication)}$
expanding, we obtain
$\begin{bmatrix}v_1\\ \vdots\\ v_m\end{bmatrix}=\underbrace{\begin{bmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&\ddots&\vdots\\ a_{m1}&\cdots&a_{mn}\end{bmatrix}}_{A\in\mathbb{R}^{m\times n}}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$. Note that we typically use a slightly different notation for the linear operator $\mathcal{A}$ and its matrix representation A. The former is a general operator without any reference to specific bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, whereas the latter is defined with respect to specific bases.

Given $\mathcal{A}$ and $v\in\mathbb{R}^m$, the following questions can be raised concerning the above set of linear equations:

1. Can we find conditions on $\mathcal{A}$ and $v\in\mathbb{R}^m$ under which at least one vector $u\in\mathbb{R}^n$ exists such that $\mathcal{A}u=v$?
2. If such vectors (solutions) exist, can we determine the number of linearly independent vectors $u\in\mathbb{R}^n$ such that $\mathcal{A}u=v$?

These questions are answered by studying the range space and nullspace of $\mathcal{A}$.

Since $\mathcal{A}:(\mathbb{R}^n,\mathbb{R})\to(\mathbb{R}^m,\mathbb{R})$,

$\mathcal{R}\{\mathcal{A}\}:=\left\{v\in(\mathbb{R}^m,\mathbb{R}):\exists u\in(\mathbb{R}^n,\mathbb{R})\ni v=\mathcal{A}u\right\}$ is a subspace of $(\mathbb{R}^m,\mathbb{R})$.

Letting $A=\begin{bmatrix}\alpha_1&\alpha_2&\cdots&\alpha_n\end{bmatrix},\ \alpha_i\in\mathbb{R}^m$, we have $v=u_1\alpha_1+u_2\alpha_2+\cdots+u_n\alpha_n$, where $u_i\in\mathbb{R}$ is the i^{th}-component of the input vector u.

Hence, $\mathcal{R}\{\mathcal{A}\}$ is the set of all linear combinations of the columns of A. But $\mathcal{R}\{\mathcal{A}\}$ is a linear space, therefore its dimension is defined and is equal to the maximum number of linearly independent vectors in $\mathcal{R}\{\mathcal{A}\}$. Thus,

$$\begin{aligned}\dim\{\mathcal{R}\{\mathcal{A}\}\} &= \text{maximum number of linearly independent columns of } A \\ &= \text{maximum number of linearly independent rows of } A \\ &= \text{largest order of all non-vanishing minors of } A\end{aligned}$$

The existence of a solution $u \in \mathbb{R}^n$ is answered by checking whether the given v is in $\mathcal{R}\{\mathcal{A}\}$. The number of solutions is found from $\mathcal{N}\{\mathcal{A}\}$ (more on this later.)

(b) The identity function $\mathcal{I}$ on $\mathcal{U}$ is a linear transformation $\mathcal{I}:\mathcal{U}\to\mathcal{V},\quad \mathcal{I}u := u$.

(c) The integral of a real-valued continuous function is a linear transformation. $\mathcal{A}: C[a,b]\to C[a,b]$

1. $(\mathcal{A}u)(t) := \int_a^t u(\tau)d\tau, \quad t\in[a,b]$

2. $(\mathcal{A}u)(t) := \int_a^t h(t,\tau)u(\tau)d\tau, \quad h(t,\tau)$ continuous on $a\le t\le b,\ a\le\tau\le b$

(d) $\mathcal{A}:\mathcal{L}_p[a,b]\to\mathcal{L}_p[a,b]$ is a linear transformation, where $\mathcal{L}_p[a,b]$ is the space of real-valued, Lebesgue-measurable functions for which the (Lebesgue) integral $\int_a^b |u(\tau)|^p\, d\tau$ exists and is finite.

$$(\mathcal{A}u)(t) := \int_a^t h(t,\tau)u(\tau)d\tau, \quad h(t,\tau) \text{ continuous on } a\le t\le b,\ a\le\tau\le b$$

for $p=2$, we get the space of finite-energy function $\mathcal{L}_2[a,b]$.

<u>Notation:</u>

Let M be a subspace of $\mathcal{V}$. Then:

$\tilde{M}$ is its set complement, i.e., $M\cup\tilde{M}=\mathcal{V}$,

M^c is its subspace complement, i.e., $M\oplus M^c=\mathcal{V}$.

Proposition:

Let $\mathcal{A}:\mathcal{U}\rightarrow\mathcal{V}$ be a linear transformation.

(a) $\mathcal{A}$ maps the zero of $\mathcal{U}$ into the zero of $\mathcal{V}$,

(b) The nullset $\mathcal{N}\{\mathcal{A}\}$, and range set $\mathcal{R}\{\mathcal{A}\}$ are always linear spaces. $\mathcal{N}\{\mathcal{A}\}$ is often denoted by $\mathrm{Ker}\{\mathcal{A}\}$ (the *kernel* of $\mathcal{A}$), and $\mathcal{R}\{\mathcal{A}\}$ is often denoted by $\mathrm{Im}\{\mathcal{A}\}$ (the *image* of $\mathcal{A}$),

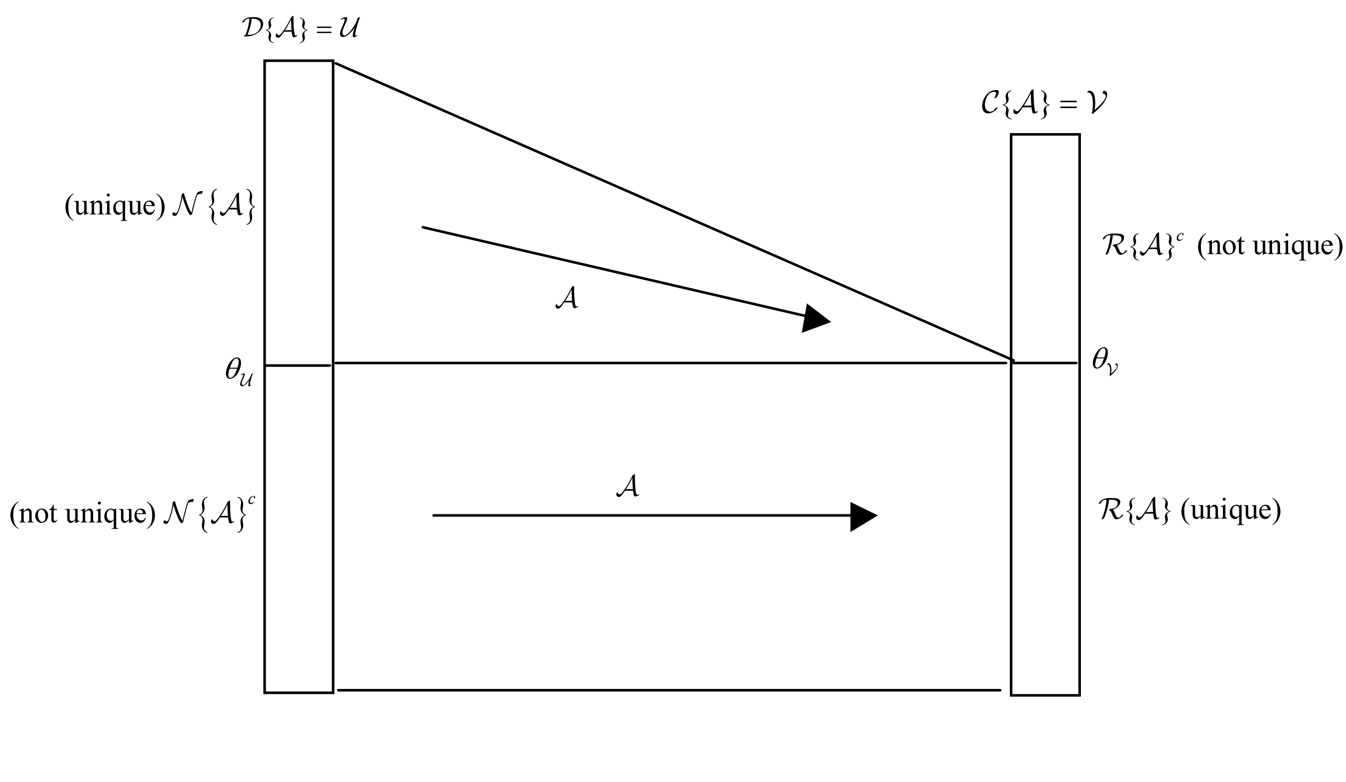
(c) The support set of $\mathcal{S}\{\mathcal{A}\}=\tilde{\mathcal{N}}\{\mathcal{A}\}$ is not a linear space (because it does not contain the zero vector),

(d) Each $\mathcal{N}\{\mathcal{A}\}^c$ is a subspace which lies entirely inside the set $\tilde{\mathcal{N}}\{\mathcal{A}\}\cup\{\theta\}$. The quotient space $\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}$ is one of the subspaces $\mathcal{N}\{\mathcal{A}\}^c$, and is unique.

2.5.2 Structure of Linear Transformations (LT)

Many of the properties of an LT, $\mathcal{A}$, are determined by the nature of $\mathcal{N}\{\mathcal{A}\}$, $\mathcal{R}\{\mathcal{A}\}$ and their complements $\mathcal{N}\{\mathcal{A}\}^c$, $\mathcal{R}\{\mathcal{A}\}^c$, respectively.

We shall represent the relationship between these spaces by the illustrative diagram shown below.



Notes:

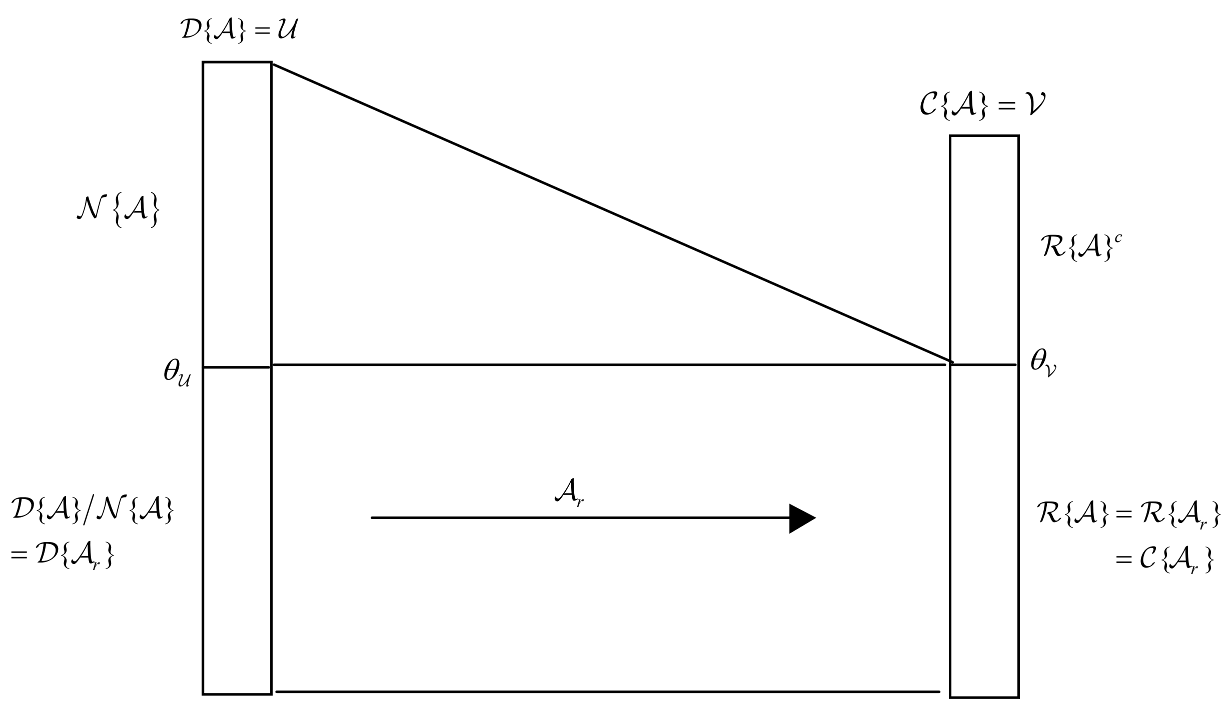
1. $\mathcal{N}\{\mathcal{A}\}^c$ is a subspace, so it includes $\theta_{\mathcal{U}}$. We generally denote such a complement as a support space $S_{sp}\{\mathcal{A}\}$.

2. $\mathcal{D}\{\mathcal{A}\}=\mathcal{N}\{\mathcal{A}\}\oplus\mathcal{N}\{\mathcal{A}\}^c$

3. $\mathcal{C}\{\mathcal{A}\}=\mathcal{R}\{\mathcal{A}\}\oplus\mathcal{R}\{\mathcal{A}\}^c$

We will show next that if the LT $\mathcal{A}$ is restricted to any of the complement subspaces $\mathcal{N}\{\mathcal{A}\}^c$, say $\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}$ in the domain $\mathcal{D}\{\mathcal{A}\}$ and to its range $\mathcal{R}\{\mathcal{A}\}$, e.g., we eliminate $\mathcal{N}\{\mathcal{A}\}$ and $\mathcal{R}\{\mathcal{A}\}^c$, then the remaining linear transformation $\mathcal{A}_r$ is onto and one-to-one. Therefore it is invertible.

Thus, $\mathcal{A}_r$ establishes a one-to-one correspondence between points in $\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}$ and points in $\mathcal{R}\{\mathcal{A}\}$, and these two subspaces have the same dimension. We show this in a series of lemmas, keeping in mind the illustrative diagram shown below.



Lemma 1:

The linear transformation $\mathcal{A}:\mathcal{U}\to\mathcal{V}$ is one-to-one iff $\mathcal{N}\{\mathcal{A}\}=\{\theta_{\mathcal{U}}\}$.

Proof:

(necessity) If $\mathcal{A}$ is one-to-one, then

$u\in\mathcal{N}\{\mathcal{A}\}\Rightarrow\mathcal{A}u=\theta_{\mathcal{V}}=\mathcal{A}\theta_{\mathcal{U}}\Rightarrow u=\theta_{\mathcal{U}}$ ($\mathcal{A}$ is one-to-one.) That is, $\mathcal{N}\{\mathcal{A}\}=\{\theta_{\mathcal{U}}\}$.

(sufficiency) If $\mathcal{N}\{\mathcal{A}\}=\{\theta_{\mathcal{U}}\}$, then

$u_1\neq u_2\Rightarrow u_1-u_2\notin\mathcal{N}\{\mathcal{A}\}\Rightarrow\mathcal{A}(u_1-u_2)\neq\theta_{\mathcal{V}}\Rightarrow\mathcal{A}u_1\neq\mathcal{A}u_2$, that is, $\mathcal{A}$ is one-to-one.

■

Let $\mathcal{A}_1$ be the D-restriction of $\mathcal{A}$ to $\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}$, i.e., discard $\mathcal{N}\{\mathcal{A}\}$.

Lemma 2:

The linear transformation $\mathcal{A}_1:\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}\to\mathcal{V}$ is

(a) one-to-one,

(b) $\mathcal{R}\{\mathcal{A}_1\}=\mathcal{R}\{\mathcal{A}\}$.

Proof:

(a) $\mathcal{N}\{\mathcal{A}_1\}=\{\theta_{\mathcal{U}}\}\Rightarrow\mathcal{A}$ is one-to-one by Lemma 1.

(b) $v\in\mathcal{R}\{\mathcal{A}\}\Rightarrow\exists u\in\mathcal{D}\{\mathcal{A}\}\ni v=\mathcal{A}u$. Decompose u into its components in the nullspace and the quotient space: $u=u_1+u_2,\quad u_1\in\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}, u_2\in\mathcal{N}\{\mathcal{A}\}$, which is possible since $\mathcal{D}\{\mathcal{A}\}=\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}\oplus\mathcal{N}\{\mathcal{A}\}$. Then, $v=\mathcal{A}(u_1+u_2)=\mathcal{A}u_1=\mathcal{A}_1u_1$. Hence, $v\in\mathcal{R}\{\mathcal{A}_1\}$. Conversely, $v\in\mathcal{R}\{\mathcal{A}_1\}\Rightarrow v\in\mathcal{R}\{\mathcal{A}\}$ (because restrictions always keep, or decrease the range.) Therefore, $\mathcal{R}\{\mathcal{A}_1\}=\mathcal{R}\{\mathcal{A}\}$. ■

Let $\mathcal{A}_r$ be the C-restriction of $\mathcal{A}_1$ to $\mathcal{R}\{\mathcal{A}\}$, i.e., discard $\mathcal{R}\{\mathcal{A}\}^c$.

Lemma 3:

The linear transformation $\mathcal{A}_r:\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}\to\mathcal{R}\{\mathcal{A}\}$ is one-to-one and onto.

Proof:

By Lemma 2(b), $\mathcal{R}\{\mathcal{A}_1\} = \mathcal{R}\{\mathcal{A}\}$, so $\mathcal{A}_r$ is the codomain restriction of $\mathcal{A}_1$ to its range. Such a restriction is always an onto transformation (Proposition in 2.3.2.) Furthermore, since $\mathcal{A}_1$ is one-to-one, so is $\mathcal{A}_r$ (Proposition in 2.3.2.)

■

Theorem: (summary)

Let $\mathcal{A}:\mathcal{U}\to\mathcal{V}$ be a linear transformation. If $\mathcal{A}_r$ is the restriction of $\mathcal{A}$ to $\mathcal{A}_r := \mathcal{A}\cap\left(\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}\times\mathcal{R}\{\mathcal{A}\}\right)$, i.e., $\mathcal{A}_r : \mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}\to\mathcal{R}\{\mathcal{A}\}$, then $\mathcal{A}_r$ is one-to-one and onto, and is therefore invertible: $\mathcal{A}_r^{-1} : \mathcal{R}\{\mathcal{A}\}\to\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}$.

Note:

If we replace $\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\}$ by any complement subspace $\mathcal{N}\{\mathcal{A}\}^c$ (or $S_{sp}\{\mathcal{A}\}$), the above theorem still holds but because $\mathcal{N}\{\mathcal{A}\}^c$ is not unique, we may obtain different inverse transformations $\mathcal{A}_r^{-1} : \mathcal{R}\{\mathcal{A}\}\to\mathcal{N}\{\mathcal{A}\}^c$.

Examples:

(a) $\mathcal{A}:\mathcal{X}\times\mathcal{Y}\to\mathcal{X}\times\mathcal{Y},\quad \mathcal{X}=\mathcal{Y}\cong\mathbb{R}$, $\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Here $\mathcal{D}\{\mathcal{A}\} = \mathcal{C}\{\mathcal{A}\} = \mathcal{X}\times\mathcal{Y}\simeq\mathbb{R}^2$.

- Nullspace $\mathcal{N}\{\mathcal{A}\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathcal{Y}$
- Range $\mathcal{R}\{\mathcal{A}\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \mathcal{X}$
- Support set $\mathcal{S}\{\mathcal{A}\} = \left\{(x,y)\in\mathcal{X}\times\mathcal{Y} : x\neq 0\right\}$
- Support space $S_{sp}\{\mathcal{A}\} = \text{span}\left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\}$ for any $a\in\mathcal{X}, b\in\mathcal{Y}$, $a\neq 0$.
- Quotient set $\mathcal{D}\{\mathcal{A}\}/\mathcal{N}\{\mathcal{A}\} = \{\text{set of lines parallel to the y-axis}\}$

- $\mathcal{A}_r$ is the restriction of $\mathcal{A}$ to any $S_{sp}\{\mathcal{A}\} \times \mathcal{R}\{\mathcal{A}\}$ is not unique, e.g., if $S_{sp}\{\mathcal{A}\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \mathcal{X}$, then $\mathcal{A}_r : \mathcal{X} \to \mathcal{X}, \quad \mathcal{A}_r x = \alpha x$ and $\mathcal{A}_r^{-1} : \mathcal{X} \to \mathcal{X}, \quad \mathcal{A}_r^{-1} x = \alpha^{-1} x$.

- $\mathcal{D}\{\mathcal{A}\} = \mathcal{D}\{\mathcal{A}\} / \mathcal{N}\{\mathcal{A}\} \oplus \mathcal{N}\{\mathcal{A}\}$, and $\mathcal{A}_r : \mathcal{D}\{\mathcal{A}\} / \mathcal{N}\{\mathcal{A}\} \to \mathcal{R}\{\mathcal{A}\}$ is one-to-one and onto. Thus, $\dim\{\mathcal{R}\{\mathcal{A}\}\} = \dim\{\mathcal{D}\{\mathcal{A}\} / \mathcal{N}\{\mathcal{A}\}\} = 1$ and $\underbrace{\dim\{\mathcal{R}\{\mathcal{A}\}\}}_{1} = \underbrace{\dim\{\mathcal{N}\{\mathcal{A}\}\}}_{1} = \underbrace{\dim\{\mathcal{D}\{\mathcal{A}\}\}}_{2}$