Lecture 5: Transformations

2.5 Transformations

2.5.1 Definition: Transformation

A transformation is a function \( \mathcal{A} : \mathcal{U} \rightarrow \mathcal{V} \) whose domain \( \mathcal{D}\{\mathcal{A}\} \) and codomain \( \mathcal{C}\{\mathcal{A}\} \) lie in vector spaces over the same field.

Such as transformation is called linear if for all \( u, u_1, u_2 \in \mathcal{D}\{\mathcal{A}\} \) and \( \alpha \in \mathcal{F} \),

\[
\mathcal{A}(u_1 + u_2) = \mathcal{A}(u_1) + \mathcal{A}(u_2) \\
\mathcal{A}(\alpha u) = \alpha \mathcal{A}(u)
\]

Examples:

(a) \( \mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad Au = v \) (matrix multiplication)

expanding, we obtain

\[
\begin{bmatrix}
v_1 \\ \\
\vdots \\ \\
v_m
\end{bmatrix} =
\begin{bmatrix}
a_{11} & \cdots & a_{1n} \\ \\
\vdots & \ddots & \vdots \\ \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
u_1 \\ \\
\vdots \\ \\
u_n
\end{bmatrix}
\]

Note that we typically use a slightly different notation for the linear operator \( \mathcal{A} \) and its matrix representation \( A \). The former is a general operator without any reference to specific bases of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), whereas the latter is defined with respect to specific bases.

Given \( \mathcal{A} \) and \( v \in \mathbb{R}^m \), the following questions can be raised concerning the above set of linear equations:

1. Can we find conditions on \( \mathcal{A} \) and \( v \in \mathbb{R}^m \) under which at least one vector \( u \in \mathbb{R}^n \) exists such that \( \mathcal{A}u = v \)?

2. If such vectors (solutions) exist, can we determine the number of linearly independent vectors \( u \in \mathbb{R}^n \) such that \( \mathcal{A}u = v \)?

These questions are answered by studying the range space and nullspace of \( \mathcal{A} \).

Since \( \mathcal{A} : (\mathbb{R}^n, \mathbb{R}) \rightarrow (\mathbb{R}^m, \mathbb{R}) \),

\[
\mathcal{R}\{\mathcal{A}\} := \{v \in (\mathbb{R}^m, \mathbb{R}) : \exists u \in (\mathbb{R}^n, \mathbb{R}) \ s.t. \ v = \mathcal{A}u\}
\]

is a subspace of \( (\mathbb{R}^m, \mathbb{R}) \).

Letting \( A = [\alpha_1, \alpha_2, \cdots, \alpha_n] \), \( \alpha_i \in \mathbb{R}^m \), we have \( v = u_1\alpha_1 + u_2\alpha_2 + \cdots + u_n\alpha_n \), where \( u_i \in \mathbb{R} \) is the \( i \)-th-component of the input vector \( u \).
Hence, \( \mathcal{R}\{A\} \) is the set of all linear combinations of the columns of \( A \). But \( \mathcal{R}\{A\} \) is a linear space, therefore its dimension is defined and is equal to the maximum number of linearly independent vectors in \( \mathcal{R}\{A\} \). Thus,

\[
\dim \{ \mathcal{R}\{A\} \} = \text{maximum number of linearly independent columns of } A = \text{maximum number of linearly independent rows of } A = \text{largest order of all non-vanishing minors of } A
\]

The existence of a solution \( u \in \mathbb{R}^n \) is answered by checking whether the given \( v \) is in \( \mathcal{R}\{A\} \). The number of solutions is found from \( \mathcal{N}\{A\} \) (more on this later.)

(b) The identity function \( \mathcal{I} \) on \( \mathcal{U} \) is a linear transformation \( \mathcal{I} : \mathcal{U} \to \mathcal{V} \), \( \mathcal{I}u := u \).

(c) The integral of a real-valued continuous function is a linear transformation. \( \mathcal{A} : C[a, b] \to C[a, b] \)

1. \( (\mathcal{A}u)(t) := \int_a^t u(\tau)d\tau, \quad t \in [a, b] \)

2. \( (\mathcal{A}u)(t) := \int_a^t h(t, \tau)u(\tau)d\tau, \quad h(t, \tau) \) continuous on \( a \leq t \leq b, \ a \leq \tau \leq b \)

(d) \( \mathcal{A} : \mathcal{L}_p[a, b] \to \mathcal{L}_p[a, b] \) is a linear transformation, where \( \mathcal{L}_p[a, b] \) is the space of real-valued, Lebesgue-measurable functions for which the (Lebesgue) integral \( \int_a^b |u(\tau)|^p d\tau \) exists and is finite.

\( (\mathcal{A}u)(t) := \int_a^t h(t, \tau)u(\tau)d\tau, \quad h(t, \tau) \) continuous on \( a \leq t \leq b, \ a \leq \tau \leq b \)

for \( p = 2 \), we get the space of finite-energy function \( \mathcal{L}_2[a, b] \).

Notation:

Let \( M \) be a subspace of \( \mathcal{V} \). Then:

\( \tilde{M} \) is its set complement, i.e., \( M \cup \tilde{M} = \mathcal{V} \),

\( M^\perp \) is its subspace complement, i.e., \( M \oplus M^\perp = \mathcal{V} \).
Proposition:

Let $A : U \rightarrow V$ be a linear transformation.

(a) $A$ maps the zero of $U$ into the zero of $V$,

(b) The nullset $\mathcal{N}\{A\}$, and range set $\mathcal{R}\{A\}$ are always linear spaces. $\mathcal{N}\{A\}$ is often denoted by $\text{Ker}\{A\}$ (the kernel of $A$), and $\mathcal{R}\{A\}$ is often denoted by $\text{Im}\{A\}$ (the image of $A$),

(c) The support set of $S\{A\} = \tilde{\mathcal{N}}\{A\}$ is not a linear space (because it does not contain the zero vector),

(d) Each $\mathcal{N}\{A\}^c$ is a subspace which lies entirely inside the set $\mathcal{N}\{A\} \cup \{\theta\}$. The quotient space $\mathcal{D}\{A\}/\mathcal{N}\{A\}$ is one of the subspaces $\mathcal{N}\{A\}^c$, and is unique.

2.5.2 Structure of Linear Transformations (LT)

Many of the properties of an LT, $A$, are determined by the nature of $\mathcal{N}\{A\}$, $\mathcal{R}\{A\}$ and their complements $\mathcal{N}\{A\}^c$, $\mathcal{R}\{A\}^c$, respectively.

We shall represent the relationship between these spaces by the illustrative diagram shown below.
Notes:

1. \( \mathcal{N}\{A\}^c \) is a subspace, so it includes \( \theta_u \). We generally denote such a complement as a support space \( S_{sp}\{A\} \).

2. \( \mathcal{D}\{A\} = \mathcal{N}\{A\} \oplus \mathcal{N}\{A\}^c \)

3. \( \mathcal{C}\{A\} = \mathcal{R}\{A\} \oplus \mathcal{R}\{A\}^c \)

We will show next that if the LT \( A \) is restricted to any of the complement subspaces \( \mathcal{N}\{A\}^c \), say \( \mathcal{D}\{A\}/\mathcal{N}\{A\} \) in the domain \( \mathcal{D}\{A\} \) and to its range \( \mathcal{R}\{A\} \), e.g., we eliminate \( \mathcal{N}\{A\} \) and \( \mathcal{R}\{A\}^c \), then the remaining linear transformation \( A_r \) is onto and one-to-one. Therefore it is invertible.

Thus, \( A_r \) establishes a one-to-one correspondence between points in \( \mathcal{D}\{A\}/\mathcal{N}\{A\} \) and points in \( \mathcal{R}\{A\} \), and these two subspaces have the same dimension. We show this in a series of lemmas, keeping in mind the illustrative diagram shown below.
Lemma 1:

The linear transformation \( A : \mathcal{U} \rightarrow \mathcal{V} \) is one-to-one iff \( \mathcal{N}\{A\} = \{\theta_i\} \).

Proof:

(necessity) If \( A \) is one-to-one, then

\( u \in \mathcal{N}\{A\} \Rightarrow Au = \theta_v = A\theta_i \Rightarrow u = \theta_i \) (\( A \) is one-to-one.) That is, \( \mathcal{N}\{A\} = \{\theta_i\} \).

(sufficiency) If \( \mathcal{N}\{A\} = \{\theta_i\} \), then

\( u_1 \neq u_2 \Rightarrow u_1 - u_2 \notin \mathcal{N}\{A\} \Rightarrow A(u_1 - u_2) \neq \theta_v \Rightarrow Au_1 \neq Au_2 \), that is, \( A \) is one-to-one. \( \blacksquare \)

Let \( A_i \) be the D-restriction of \( A \) to \( \mathcal{D}\{A\}/\mathcal{N}\{A\} \), i.e., discard \( \mathcal{N}\{A\} \).

Lemma 2:

The linear transformation \( A_i : \mathcal{D}\{A\}/\mathcal{N}\{A\} \rightarrow \mathcal{V} \) is

(a) one-to-one,

(b) \( \mathcal{R}\{A_i\} = \mathcal{R}\{A\} \).

Proof:

(a) \( \mathcal{N}\{A_i\} = \{\theta_i\} \Rightarrow A \) is one-to-one by Lemma 1.

(b) \( v \in \mathcal{R}\{A\} \Rightarrow \exists u \in \mathcal{D}\{A\} \ni v = Au \). Decompose \( u \) into its components in the nullspace and the quotient space: \( u = u_1 + u_2 \), \( u_i \in \mathcal{D}\{A\}/\mathcal{N}\{A\} \), \( u_2 \in \mathcal{N}\{A\} \), which is possible since \( \mathcal{D}\{A\} = \mathcal{D}\{A\}/\mathcal{N}\{A\} \oplus \mathcal{N}\{A\} \). Then, \( v = A(u_1 + u_2) = Au_i = A_iu_i \). Hence, \( v \in \mathcal{R}\{A_i\} \). Conversely, \( v \in \mathcal{R}\{A_i\} \Rightarrow v \in \mathcal{R}\{A\} \) (because restrictions always keep, or decrease the range.) Therefore, \( \mathcal{R}\{A_i\} = \mathcal{R}\{A\} \). \( \blacksquare \)

Let \( A_i \) be the C-restriction of \( A_i \) to \( \mathcal{R}\{A\} \), i.e., discard \( \mathcal{R}^c\{A\} \).

Lemma 3:

The linear transformation \( A_i : \mathcal{D}\{A\}/\mathcal{N}\{A\} \rightarrow \mathcal{R}\{A\} \) is one-to-one and onto.
Proof:

By Lemma 2(b), $R\{A_1\} = R\{A\}$, so $A_r$ is the codomain restriction of $A$, to its range. Such a restriction is always an onto transformation (Proposition in 2.3.2.). Furthermore, since $A_r$ is one-to-one, so is $A_r$ (Proposition in 2.3.2.).

Theorem: (summary)

Let $A : U \rightarrow V$ be a linear transformation. If $A_r$ is the restriction of $A$ to $A_r := A \cap (D\{A\}/N\{A\} \times R\{A\})$, i.e., $A_r : D\{A\}/N\{A\} \rightarrow R\{A\}$, then $A_r$ is one-to-one and onto, and is therefore invertible: $A_r^{-1} : R\{A\} \rightarrow D\{A\}/N\{A\}$.

Note:

If we replace $D\{A\}/N\{A\}$ by any complement subspace $N\{A\}^c$ (or $S_{sp}\{A\}$), the above theorem still holds but because $N\{A\}^c$ is not unique, we may obtain different inverse transformations $A_r^{-1} : R\{A\} \rightarrow N\{A\}^c$.

Examples:

(a) $A : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y}$, $\mathcal{X} = \mathcal{Y} \cong \mathbb{R}$, $A:\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Here $D\{A\} = C\{A\} = \mathcal{X} \times \mathcal{Y} = \mathbb{R}^2$.

- Nullspace $N\{A\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathcal{Y}$
- Range $R\{A\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \mathcal{X}$
- Support set $S\{A\} = \left\{ (x, y) \in \mathcal{X} \times \mathcal{Y} : x \neq 0 \right\}$
- Support space $S_{sp}\{A\} = \text{span} \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \right\}$ for any $a \in \mathcal{X}$, $b \in \mathcal{Y}$, $a \neq 0$.
- Quotient set $D\{A\}/N\{A\} = \{\text{set of lines parallel to the y-axis}\}$
• \( \mathcal{A}_r \) is the restriction of \( \mathcal{A} \) to any \( S_{sp}(\mathcal{A}) \times \mathcal{R}(\mathcal{A}) \) is not unique, e.g., if 
\[ S_{sp}(\mathcal{A}) = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathcal{X}, \]
then \( \mathcal{A}_r : \mathcal{X} \to \mathcal{X}, \mathcal{A}_r x = \alpha x \) and 
\( \mathcal{A}_r^{-1} : \mathcal{X} \to \mathcal{X}, \mathcal{A}_r^{-1} x = \alpha^{-1} x. \)

• \( \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})/\mathcal{N}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A}), \) and \( \mathcal{A}_r : \mathcal{D}(\mathcal{A})/\mathcal{N}(\mathcal{A}) \to \mathcal{R}(\mathcal{A}) \) is one-to-one and onto. Thus, \( \dim \{ \mathcal{R}(\mathcal{A}) \} = \dim \{ \mathcal{D}(\mathcal{A})/\mathcal{N}(\mathcal{A}) \} = 1 \) and 
\[ \underbrace{\dim \{ \mathcal{R}(\mathcal{A}) \}}_{1} = \underbrace{\dim \{ \mathcal{N}(\mathcal{A}) \}}_{1} = \underbrace{\dim \{ \mathcal{D}(\mathcal{A}) \}}_{2} \]