2.2 Properties of Functions

2.2.1 Composition of functions

Let \( f : \mathcal{U} \to \mathcal{B} \) be a function with domain \( \mathcal{D}(f) = \mathcal{U} \subseteq \mathcal{A} \) and range \( \mathcal{R}(f) \subseteq \mathcal{B} \), and let \( g : \mathcal{V} \to \mathcal{C} \) be a function with domain \( \mathcal{D}(g) = \mathcal{V} \subseteq \mathcal{B} \) and range \( \mathcal{R}(g) \subseteq \mathcal{C} \).

Definition: Composition

The composition of \( g \) following \( f \), denoted by \( g \circ f \), is the function from \( \mathcal{A} \) to \( \mathcal{C} \) given by:

\[
g \circ f := \{(a, c) \in \mathcal{A} \times \mathcal{C} : \exists b \in \mathcal{B} \; \exists (a, b) \in f, (b, c) \in g\}
\]

The domain of \( g \circ f \) is given by:

\[
\mathcal{D}(g \circ f) := \{u \in \mathcal{D}(f) : f(u) \in \mathcal{D}(g)\}
\]

The range of \( g \circ f \) is given by:

\[
\mathcal{R}(g \circ f) := \{g(f(u)) : u \in \mathcal{D}(g \circ f)\}
\]
Note that other equivalent definitions may be obtained.

Examples:

(a) Consider the real-valued functions:

\[ f : \mathbb{R} \to \mathbb{R}, \quad f(x) := 2x \]
\[ g : \mathbb{R} \to \mathbb{R}, \quad g(x) := 3x^2 - 1 \]

Since \( \mathcal{D}\{g\} \) is the set of all real numbers \( \mathbb{R} \), and \( \mathcal{R}\{f\} = \mathbb{R} \subseteq \mathcal{D}\{g\} \), then \( \mathcal{D}\{g \circ f\} = \mathbb{R} \) and the composition is

\[ g \circ f : \mathbb{R} \to \mathbb{R}, \quad g \circ f(x) := 12x^2 - 1. \]

Exercise: find the domain \( \mathcal{D}\{f \circ g\} \), and the ranges \( \mathcal{R}\{f \circ g\}, \mathcal{R}\{g \circ f\} \).

(b) Consider the real-valued functions:

\[ f : \mathbb{R} \to \mathbb{R}, \quad f(x) := 2x \]
\[ h : \{x \in \mathbb{R} : x \geq 1\} \to \mathbb{R}, \quad h(x) := \sqrt{x-1} \]

The domain of \( h \circ f \) is \( \mathcal{D}\{h \circ f\} = \left\{ x \in \mathbb{R} : x \geq \frac{1}{2} \right\} \) and

\[ h \circ f : \mathcal{D}\{h \circ f\} \to \mathbb{R}, \quad h \circ f(x) := \sqrt{2x-1}. \]

Exercise: find \( \mathcal{D}\{f \circ h\}, \mathcal{R}\{f \circ h\} \) and the expression for \( f \circ h(x) \).

### 2.2.2 Injective (one-to-one) and Surjective (onto) Functions

Consider the function \( f : \mathcal{U} \to \mathcal{V} \).

**Definition: Injective (one-to-one)**

\( f \) is said to be *injective* or *one-to-one* if, given \( u_1, u_2 \in \mathcal{U} \),

\[ f(u_1) = f(u_2) \iff u_1 = u_2 \]

In words: any element in the range of \( f \) has one and only one associated element in the domain.

**Definition: Surjective (onto)**

\( f \) is said to be *surjective* or *onto* if its range and its codomain are equal, i.e.,

\[ \mathcal{R}\{f\} = \mathcal{C}\{f\} = \mathcal{V} \]
Definition: Bijective

$f$ is said to be bijective if it is one-to-one and onto.

Examples:

(a) $x^2 : \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$

$\mathcal{D}(f) = \mathcal{C}(f) = \mathbb{R}, \ \mathcal{R}(f) = [0, +\infty)$.

This function is neither one-to-one as $(-a)^2 = (a)^2, \ a \in \mathcal{D}(f)$, nor onto as $\mathcal{R}(f) \neq \mathbb{R}$.

(b) $x^3 : \mathbb{R} \to \mathbb{R}, \ x \mapsto x^3$

$\mathcal{D}(f) = \mathcal{C}(f) = \mathbb{R}, \ \mathcal{R}(f) = \mathbb{R}$.

This function is one-to-one as should be clear from the graph in the figure below, and onto as $\mathcal{R}(f) = \mathbb{R}$.

Definition: Identity function

The identity function $\mathcal{I}$ on $\mathcal{U}$ is the function

$$\mathcal{I} : \mathcal{U} \to \mathcal{V}, \ \mathcal{I}u := u.$$

Definition: Inverse of a function

The inverse $f^{-1}$ of function $f : \mathcal{U} \to \mathcal{V}$ is the function $f^{-1} : \mathcal{V} \to \mathcal{U}$ satisfying the conditions:
1. \( f \circ f^{-1} = I \) on \( V \)

2. \( f^{-1} \circ f = I \) on \( U \)

That is, \( f \circ f^{-1}(v) = v, \forall v \in V \) and \( f^{-1} \circ f(u) = u, \forall u \in U \).

**Proposition:**

(a) If \( f^{-1} \) exists, then \( f^{-1} = \{(v, u) : (u, v) \in f\} \).

Example: for a real-valued function of a real variable, the inverse is the reflection of the graph of \( f \) about the 45° line.

(b) If \( f \) has an inverse \( f^{-1} \), then \( f^{-1} \) is unique.

(c) \( f^{-1} \) exists iff \( f \) is one-to-one and onto (bijective).

**Proof:**

(a) Suppose \( f^{-1} : V \to U \) exists. Then, from the definition of the inverse, \( f^{-1} \circ f(u) = u, \forall u \in U \). But from the definition of a composition of functions, we have \( f^{-1} \circ f = \{(u, u) \in U \times U : \exists v \in V \exists (u, v) \in f, (v, u) \in f^{-1}\} \), and hence:

\( \forall u \in U, \exists v \in V \exists (u, v) \in f \) and \((v, u) \in f^{-1}\). There still remains the possibility that for some \( v \in V \), \((v, u) \in f^{-1} \) but \((u, v) \notin f \). However, an argument similar to the one we used above but for \( f \circ f^{-1} : V \to V \) leads to: \( \forall v \in V, \exists u \in U \exists (u, v) \in f \) and \((v, u) \in f^{-1}\).

We conclude that \( f^{-1} = \{(v, u) : (u, v) \in f\} \).

(b) We prove this assertion by contradiction. Suppose \( f \) has two inverses \( f_1^{-1} \neq f_2^{-1} \). Then from the definition of the inverse \( f_1^{-1} \circ f(u) = f_2^{-1} \circ f(u) = u, \forall u \in U \). This implies that \( \forall u \in U, \exists v \in V \exists (u, v) \in f_1^{-1} \) and \((v, u) \in f_2^{-1}\). Some elements of the graphs could still be different. Suppose that \( \exists v_0 \in V \exists (v_0, u) \in f_1^{-1} \), but \((v_0, u) \notin f_2^{-1}\). Then \( f_1^{-1}(f(u)) = f_1^{-1}(v_0) = u, \) but \( f_2^{-1}(f(u)) = f_2^{-1}(v_0) \neq u \), a contradiction with the assumption that \( f_2^{-1} \) is an inverse. Therefore, \( f_1^{-1} = f_2^{-1} = f^{-1} \) is unique.
(c) Suppose that $f$ is not one-to-one. Then $\exists u_1 \neq u_2 \ni f(u_1) = f(u_2) = v$. Obviously, there can't be a function called "inverse" that would give back both $u_1$ and $u_2$ (a function is single-valued.) Thus $f$ must be one-to-one. Now assume $f : \mathcal{U} \to \mathcal{V}$ is not onto. Then $\exists v \in \mathcal{V} \ni v \not\in \mathcal{R}(f) = \mathcal{D}(f^{-1})$, and hence $f \circ f^{-1} = \mathcal{I}$ on $\mathcal{V}$ is impossible to obtain. Thus $f$ must be onto. So far, we have shown that $f$ must be one-to-one and onto to have an inverse. Finally, we have to show that this is also a sufficient condition. Suppose that $f$ is one-to-one and onto. Then, $\mathcal{V} = \mathcal{R}(f)$ (onto), and $f(u_1) = f(u_2) = v \iff u_1 = u_2$ (one-to-one). Define the set $g = \{(v,u) : (u,v) \in f^1\}$. This graph is that of a function $g : \mathcal{V} \to \mathcal{U}$, with $\mathcal{V} = \mathcal{D}(g) = \mathcal{R}(f)$, and for which $g \circ f = \mathcal{I}$ on $\mathcal{U}$. Moreover, $f \circ g = \mathcal{I}$ on $\mathcal{V}$ since $g(v_1) = g(v_2) = u \iff v_1 = v_2$. Therefore $g = f^{-1}$.

\[\square\]

2.3 Restrictions and extensions of functions

2.3.1 Definition of a restriction of a function

A restriction $f'$ of any function $f$ is a function whose graph is a subset of the graph of $f$.

Example: Let $f : \mathcal{U} \to \mathcal{V}$ have the graph shown below.

\[\begin{align*}
&\mathcal{U} \\
\downarrow &\uparrow \\
&\mathcal{V} \\
\downarrow &\uparrow \\
&f \\
\downarrow &\uparrow \\
&f' \\
\downarrow &\uparrow \\
&\mathcal{U}' \\
\end{align*}\]

A D-restriction (stands for domain restriction) $f' : \mathcal{U}' \to \mathcal{V}$ of the function $f : \mathcal{U} \to \mathcal{V}$ where $\mathcal{U'} \subset \mathcal{U}$ is the function $f' = f \cap (\mathcal{U}' \times \mathcal{V})$. 

"
A C-restriction (stands for codomain restriction) $f'$ of the function $f : \mathcal{U} \rightarrow \mathcal{V}$ to a subset $\mathcal{V}' \subset \mathcal{V}$ is the set $f' = f \cap (\mathcal{U} \times \mathcal{V}')$. Note that $f'$ is not necessarily a function as it may be undefined for some $u \in \mathcal{U}$.

### 2.3.2 Definition of an extension of a function

If $g$ is a function with domain $\mathcal{D}\{g\}$ and we have a set $\mathcal{D}' \supset \mathcal{D}\{g\}$, then any function $g'$ with domain $\mathcal{D}'$ such that

$$g'(u) = g(u), \quad \forall u \in \mathcal{D}\{g\}$$

is an extension of $g$ to domain $\mathcal{D}'$.

**Proposition:**

A codomain restriction of a function $f : \mathcal{U} \rightarrow \mathcal{V}$ to a subset $\mathcal{V}' \subset \mathcal{V}$ is a function iff $\mathcal{V}'$ contains the range of $f$, and is onto iff $\mathcal{V}' = R\{f\}$.

**Proposition:** The C-restriction or D-restriction of any one-to-one function is one-to-one.

### 2.4 Null and Support Sets

Assume we have the function $f : \mathcal{U} \rightarrow \mathcal{V}$ where $\mathcal{V}$ has a zero element.

#### 2.4.1 Definition: Null Set

The null set of $f : \mathcal{U} \rightarrow \mathcal{V}$ is the set $\mathcal{N}\{f\} := \{u \in \mathcal{U} : f(u) = \theta\}$.

#### 2.4.2 Definition: Support Set

The support set of $f : \mathcal{U} \rightarrow \mathcal{V}$ is the set $\mathcal{S}_{sp}\{f\} := \{u \in \mathcal{U} : f(u) \neq \theta\}$. Note that the domain of $f$ is the union of its null set and its support set, i.e., $\mathcal{D}\{f\} = \mathcal{S}_{sp}\{f\} \cup \mathcal{N}\{f\}$.

**Example:**

![Diagram](image-url)
In systems theory, we think of the domain and codomain of a function (system) as its sets of inputs and outputs, respectively. The range represents the outputs that are attainable by the system.