# *Lecture 20: Riccati Equations and Least Squares Feedback Control*

## 5.6.4 State Feedback via Riccati Equations

A recursive approach in generating the matrix-valued function $W^{-1}(t,t_1)$ is to find a differential equation for it, for the purpose of realizing the optimal control law in real time.

Thus, the matrix differential equation can be approximated by a difference equation which can be solved numerically.

Theorem: (properties of the backward controllability Grammian)

The backward controllability Grammian $W(t_0,t_1)$ has the following properties for $t_0 \le t \le t_1$:

(a) $W(t,t_1)$ is symmetric,

(b) $W(t,t_1)$ is positive semidefinite,

(c) $\frac{d}{dt}W(t,t_1) = A(t)W(t,t_1) + W(t,t_1)A^*(t) - B(t)B^*(t), \; W(t_1,t_1) = \theta_{n\times n}$

(d) $W(t_0,t_1)$ satisfies the equation $W(t_0,t_1) = W(t_0,t) + \Phi(t_0,t)W(t,t_1)\Phi^*(t_0,t)$

Proof:

Similar to Theorem on properties of the forward controllability Grammian in L18.

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Note:

The forward and backward controllability Grammians are related as follows:

$$\begin{aligned} W(t,t_1) &= \Phi(t,t_1)M(t,t_1)\Phi^*(t,t_1) \\ M(t,t_1) &= \Phi(t_1,t)W(t,t_1)\Phi^*(t_1,t) \end{aligned} \tag{5.72}$$

Proposition:

The inverse of the backward controllability Grammian $W^{-1}(t,t_1)$ satisfies the matrix differential Riccati equation:

$$\begin{aligned} &\frac{d}{dt}W^{-1}(t,t_1) = -A(t)W^{-1}(t,t_1) - W^{-1}(t,t_1)A^*(t) + W^{-1}(t,t_1)B(t)B^*(t)W^{-1}(t,t_1), \\ &W^{-1}(t_1,t_1) = \infty \end{aligned} \tag{5.73}$$

Proof:

Differentiate: $\frac{d}{dt}W^{-1}(t,t_1) = -W^{-1}(t,t_1)\frac{d}{dt}W(t,t_1)W^{-1}(t,t_1)\ldots$

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Remarks:

(a) One way of generating $W^{-1}(t,t_1)$ in real time is to solve recursively a difference equation approximating the differential Riccati Equation (5.73), subject to the initial condition $W(t_0,t_1)$.

(b) The optimal control for the transfer to the origin $(x_0,t_0) \to (0,t_1)$ is the state feedback $u_{opt}(t) = -B(t)^* W^{-1}(t,t_1)x(t)$.

## 5.7 Least Squares State Feedback Control

The least-squares feedback control problem is defined as follows. Given:

- A system $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ \ x(t_0) = x_0$,
- A set $\Sigma_{x(t_1)}$ to which $x(t_1)$ belongs,
- A cost function $J_{t_0,t_1}(x_0,u_{[t_0,t_1]}) = \int_{t_0}^{t_1} \begin{bmatrix} u^*(t) & x^*(t) \end{bmatrix} \begin{bmatrix} I & N(t) \\ N^*(t) & L(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt + x^*(t_1)Qx(t_1)$

The least squares problem is that of finding a control law $u \in \mathcal{U}_{ad}$ such that the cost $J_{t_0,t_1}(x_0,u_{[t_0,t_1]})$ is minimized subject to the constraints.

This problem was partly solved for the minimum-norm input, but here we add the state in the cost function in order to minimize it as well. The least-squares problem is classified into:

(i) Free-end-point problem,

(ii) Fixed-end-point problem.

### 5.7.1 Free-End-Point Problem

The free-end-point problem is defined as follows. Given:

- A system $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ \ x(t_0) = x_0$,

- A cost function $J_{t_0,t_1}(x_0,u_{[t_0,t_1]}) = \int_{t_0}^{t_1}\left[u^*(t)u(t)+x^*(t)L(t)x(t)\right]dt + x^*(t_1)Qx(t_1)$,

  $L(t)=L^*(t),\ Q=Q^*$ ( $N(t)=\theta$ in the general problem definition)

Find a control law $u \in \mathcal{U}_{ad}$ such that the cost $J_{t_0,t_1}(x_0,u_{[t_0,t_1]})$ is minimized subject to the constraints.

Note that because $L(t)$ and $Q$ are not assumed to be nonnegative it is not generally true that a minimum exists for $J_{t_0,t_1}(x_0,u_{[t_0,t_1]})$. However, in many control applications these conditions will hold.

It turns out that the determination of conditions under which a minimum exists as well as the actual calculation of the minimizing control can be made to depend upon the solution of the following Riccati equation:

$$\dot{P}(t) = -A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - L(t) \qquad (5.74)$$

Since this equation is nonlinear, it is not clear that for a given initial condition $P_0$, a solution will exist. Moreover, even if solutions do exist for some time intervals, they may fail to exist over longer intervals because of the possibility of having a finite escape time.

<u>Example:</u> (finite escape time)

Suppose that the Riccati equation is scalar and has the form

$$\begin{aligned} &\dot{p}(t) = p^2(t)+1,\quad p(0)=0 \\ &\Rightarrow p(t) = \tan(t) \\ &\Rightarrow p(\pi/2) = +\infty \end{aligned}$$

Thus, $p(t)$ does not exist over the interval $0 \le t \le 2$.

<u>Notation:</u>

The solution to the Riccati equation (5.74) will be denoted as $\Pi(t;P_1,t_1)$ (or $P(t)$), meaning that its value at time $t_1$ is $\Pi(t_1;P_1,t_1)=P_1$, and its value at time $t$ is generated from $\Pi(t_1;P_1,t_1)=P_1$.

<u>Lemma:</u>

Suppose $P(t)=P^*(t)$ and $\dot{P}(t)$ on the interval $t\in[t_0,t_1]$. Then for the system $\dot{x}(t)=A(t)x(t)+B(t)u(t)$, we have:

$$\int_{t_0}^{t_1}\begin{bmatrix}u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix}0 & B^*(t)P(t)\\ P(t)B(t) & \dot{P}(t)+A^*(t)P(t)+P(t)A(t)\end{bmatrix}\begin{bmatrix}u(t)\\ x(t)\end{bmatrix}dt - x^*(t)P(t)x(t)\Big|_{t_0}^{t_1} = 0.$$

Proof:

If $x(\cdot)$ is a differentiable trajectory and if $P(\cdot)$ is a differentiable matrix-valued function of time, then:

$$\int_{t_0}^{t_1} \frac{d}{dt}\left[x^*(t)P(t)x(t)\right]dt = x^*(t)P(t)x(t)\Big|_{t_0}^{t_1}$$

$\Rightarrow$

$$x^*(t)P(t)x(t)\Big|_{t_0}^{t_1} = \int_{t_0}^{t_1}\left[\dot{x}^*(t)P(t)x(t) + x^*(t)\dot{P}(t)x(t) + x^*(t)P(t)\dot{x}(t)\right]dt$$

$$= \int_{t_0}^{t_1}\left[\begin{array}{l}\left\{A(t)x(t)+B(t)u(t)\right\}^* P(t)x(t) + \\ x^*(t)\left\{-A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - L(t)\right\}x(t) + x^*(t)P(t)\left\{A(t)x(t)+B(t)u(t)\right\}\end{array}\right]dt$$

$$= \int_{t_0}^{t_1}\left[u^*(t) \quad x^*(t)\right]\begin{bmatrix} 0 & B^*(t)P(t) \\ P(t)B(t) & \dot{P}(t) + A^*(t)P(t) + P(t)A(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt$$

■

Theorem:

Suppose $L(t) = L^*(t),\ Q = Q^*$ and there exists over $[t_0, t_1]$ a solution to the Riccati differential equation $\dot{P}(t) = -A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - L(t), \quad P(t_1) = Q$, then:

(a) There exists a control $u(\cdot)$ which minimizes $J_{t_0,t_1}(x_0, u_{opt[t_0,t_1]})$ subject to the constraints $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ \ x(t_0) = x_0$,

(b) The minimum value of the cost function is $J_{t_0,t_1}(x_0, u_{opt[t_0,t_1]}) = x_0^{\ *}\Pi(t_0; Q, t_1)x_0$

(c) The optimal feedback control law is $u_{opt}(t) = -B(t)^*\Pi(t; Q, t_1)x(t)$

(d) The optimal open-loop control law is $u_{opt[t_0,t_1]}(t) = -B(t)^*\Pi(t; Q, t_1)\Phi(t, t_0)x_0$ where $\Phi(t, t_0)$ is the state transition matrix for $\dot{x}(t) = \left[A(t) - B(t)B^*(t)\Pi(t; Q, t_1)\right]x(t)$.

Proof:

To ease the notation, let's use $P(t)$ instead of $\Pi(t; Q, t_1)$ to denote the solution of the Riccati equation. From the lemma, we have the identity for $x^*(t)P(t)x(t)\Big|_{t_0}^{t_1}$, which we use in the cost function:

$$
\begin{aligned}
&J_{t_0,t_1}(x_0,u_{[t_0,t_1]})\\
&=\int_{t_0}^{t_1}\begin{bmatrix} u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & L(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt + x^*(t_1)Qx(t_1)\\
&=\int_{t_0}^{t_1}\begin{bmatrix} u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & L(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt + x^*(t)P(t)x(t)\Big|_{t_0}^{t_1} + x^*(t_0)P(t_0)x(t_0)\\
&=\int_{t_0}^{t_1}\begin{bmatrix} u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & L(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt\\
&+\int_{t_0}^{t_1}\begin{bmatrix} u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix} 0 & B^*(t)P(t) \\ P(t)B(t) & \dot{P}(t)+A^*(t)P(t)+P(t)A(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt + x^*(t_0)P(t_0)x(t_0)\\
&=\int_{t_0}^{t_1}\begin{bmatrix} u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix} I & B^*(t)P(t) \\ P(t)B(t) & \dot{P}(t)+A^*(t)P(t)+P(t)A(t)+L(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt + x^*(t_0)P(t_0)x(t_0)\\
&=\int_{t_0}^{t_1}\begin{bmatrix} u^*(t) & x^*(t)\end{bmatrix}\begin{bmatrix} I & B^*(t)P(t) \\ P(t)B(t) & P(t)B(t)B^*(t)P(t)\end{bmatrix}\begin{bmatrix} u(t) \\ x(t)\end{bmatrix}dt + x^*(t_0)P(t_0)x(t_0)\\
&=\int_{t_0}^{t_1}\left\|u(t)+B^*(t)P(t)x(t)\right\|^2 dt + x^*(t_0)P(t_0)x(t_0)
\end{aligned}
$$

From this last equality, we conclude that the input that minimizes the cost function is $u_{opt}(t)=-B^*(t)P(t)x(t)$ which proves (a) and (c), and for which the cost function reaches its minimum $J_{t_0,t_1}(x_0,u_{opt[t_0,t_1]})=x_0{}^*\Pi(t_0;Q,t_1)x_0$, proving (b). For (d), notice that the optimal trajectory $x_{opt}$ is given by the closed-loop equation:

$$\dot{x}_{opt}(t)=\left[A(t)-B(t)B^*(t)\Pi(t;Q,t_1)\right]x_{opt}(t).$$

Thus, if $\Phi(t,t_0)$ is the state transition matrix for the closed-loop system, we have $x_{opt}(t)=\Phi(t,t_0)x(t_0)$ and hence $u_{opt}(t)=-B^*(t)P(t)\Phi(t,t_0)x(t_0)$ is the optimal open-loop policy.

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Remarks:

Suppose that $L(t)=0$, which amounts to weighting only the control signal in the cost function. Then the Riccati equation reduces to

$$\dot{P}(t)=-A^*(t)P(t)-P(t)A(t)+P(t)B(t)B^*(t)P(t).$$

Hence, $P(\cdot)$ satisfies the same equation as the inverse of the backward controllability Grammian $W^{-1}(t,t_1)$. Furthermore, it can be shown that $\Pi(t;Q,t_1) = \left[W(t,t_1) + \Phi(t,t_1)Q^{-1}\Phi^*(t,t_1)\right]^{-1}$ provided the inverse exists.

Theorem:

Assume $Q = Q^*$. Let $W(t,t_1)$ be the backward controllability Grammian of $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ \ x(t_0) = x_0$. If the matrix $W(t,t_1) + \Phi(t,t_1)Q^{-1}\Phi^*(t,t_1)$ is invertible, $\forall t \in [t_0, t_1]$, then there exists a control that minimizes

$$J_{t_0,t_1}(x_0, u_{[t_0,t_1]}) = \int_{t_0}^{t_1} u^*(t)u(t)dt + x^*(t_1)Qx(t_1).$$

Proof:

Follows from the previous remark.

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### 5.7.2 Fixed-End-Point Problem

The fixed-end-point problem is defined as follows. Given:

- A system $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ \ x(t_0) = x_0,\ \ x(t_1) = x_1$,
- A cost function $J_{t_0,t_1}(x_0, u_{[t_0,t_1]}) = \int_{t_0}^{t_1} \left[u^*(t)u(t) + x^*(t)L(t)x(t)\right]dt$, $L(t) = L^*(t),\ \ Q = Q^*$ ($N(t) = \theta$ in the general problem definition)

Find a control law $u \in \mathcal{U}_{ad}$ such that the cost $J_{t_0,t_1}(x_0, u_{[t_0,t_1]})$ is minimized subject to the constraints.

This is an extension of the controllability problem studied earlier in which a weight on the state is added.

However, here we discuss the solution from the optimal control point of view. If the terminal state is required to belong to a certain set instead of a fixed vector, the methodology can be modified to account for this requirement. In general, *calculus of variations* is the subject that deals with such optimization problems.

From Section 5.6, we found an expression for the minimum-norm control:

$$u_{opt}(t) = B(t)^*\Phi^*(t_0,t)W^{-1}(t_0,t_1)\left[\Phi(t_0,t_1)x_1 - x_0\right] \qquad (5.75)$$

which leads to the following lemma.

Lemma:

Let $W(t_0, t_1)$ be the backward controllability Grammian of $\dot{x}(t) = A(t)x(t) + B(t)u(t)$. If $u_{opt}(t)$ is any control of the form $u_{opt}(t) = -B(t)^* \Phi(t_0, t)^* \xi$ where vector $\xi$ is a solution of

$$W(t_0, t_1)\xi = \left[\Phi(t_0, t_1)x_1 - x_0\right], \tag{5.76}$$

then the control $u_{opt}(t)$ drives the state from $(x_0, t_0) \to (x_1, t_1)$. If $u_1(t)$ is any other control with the same property, i.e., $(x_0, t_0) \to (x_1, t_1)$, then

$$\left\|u_{opt}\right\|_{\mathcal{L}_2}^2 = \int_{t_0}^{t_1} u_{opt}^*(t)u_{opt}(t)dt \le \int_{t_0}^{t_1} u_1^*(t)u_1(t)dt\ . \tag{5.77}$$

Moreover, if $W(t_0, t_1)$ is nonsingular, then

$$\left\|u_{opt}\right\|_{\mathcal{L}_2[t_0,t_1]}^2 = \int_{t_0}^{t_1} u_{opt}^*(t)u_{opt}(t)dt = \left[x_0 - \Phi(t_0, t_1)x_1\right]^* W^{-1}(t_0, t_1)\left[x_0 - \Phi(t_0, t_1)x_1\right]. \tag{5.78}$$

Theorem:

Suppose that $L(t) = L(t)^*$, and that there exists a symmetric matrix $P(t_1)$ such that the solution $\Pi(t; P(t_1), t_1)$ of the differential Riccati equation

$$\dot{P}(t) = -A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - L(t) \tag{5.79}$$

holds. Then,

(a) A differentiable trajectory $x(\cdot)$ defined on the interval $[t_0, t_1]$ and satisfying $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ \ x(t_0) = x_0,\ \ x(t_1) = x_1$ minimizes the cost

$$J_{t_0,t_1}(x_0, u_{[t_0,t_1]}) = \int_{t_0}^{t_1} \left[u^*(t)u(t) + x^*(t)L(t)x(t)\right]dt \tag{5.80}$$

if and only if it minimizes the costs

$$\tilde{J}_{t_0,t_1}(x_0, u_{[t_0,t_1]}) = \int_{t_0}^{t_1} v^*(t)v(t)dt = \left\|v\right\|_{\mathcal{L}_2[t_0,t_1]}^2 \tag{5.81}$$

subject to the constraints

$$\dot{x}(t) = \left[A(t) - B(t)B^*(t)\Pi(t; P(t_1), t_1)\right]x(t) + B(t)v(t),\ \ x(t_0) = x_0,\ \ x(t_1) = x_1 \tag{5.82}$$

(b) Along any state trajectory satisfying the boundary conditions, we have

$$J_{t_0,t_1}(x_0,u_{[t_0,t_1]}) = \tilde{J}_{t_0,t_1}(x_0,u_{[t_0,t_1]}) + x_0^{\ *}\Pi(t_0;P(t_1),t_1)x_0 - x_1^{\ *}P(t_1)x_1 \tag{5.83}$$

<u>Proof:</u>

From the Lemma in 5.7.1, we have

$$\begin{aligned}
&J_{t_0,t_1}(x_0,u_{[t_0,t_1]}) \\
&= \int_{t_0}^{t_1} \begin{bmatrix} u^*(t) & x^*(t) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & L(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt \\
&= \int_{t_0}^{t_1} \begin{bmatrix} u^*(t) & x^*(t) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & L(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt \\
&+ \int_{t_0}^{t_1} \begin{bmatrix} u^*(t) & x^*(t) \end{bmatrix} \begin{bmatrix} 0 & B^*(t)P(t) \\ P(t)B(t) & \dot{P}(t) + A^*(t)P(t) + P(t)A(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt - x^*(t)P(t)x(t)\Big|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \begin{bmatrix} u^*(t) & x^*(t) \end{bmatrix} \begin{bmatrix} I & B^*(t)P(t) \\ P(t)B(t) & \dot{P}(t) + A^*(t)P(t) + P(t)A(t) + L(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt - x^*(t)P(t)x(t)\Big|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \begin{bmatrix} u^*(t) & x^*(t) \end{bmatrix} \begin{bmatrix} I & B^*(t)P(t) \\ P(t)B(t) & P(t)B(t)B^*(t)P(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt - x^*(t)P(t)x(t)\Big|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \left\| u(t) + B^*(t)P(t)x(t) \right\|^2 dt + x_0^{\ *}P(t_0)x_0 - x_1^{\ *}P(t_1)x_1
\end{aligned}$$

Letting $v(t) := u(t) + B^*(t)P(t)x(t)$ (which proves (b)), we obtain the equivalent problem of minimizing the norm of $v$ subject to the constraint

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)\left[ v(t) - B^*(t)P(t)x(t) \right] \\
&= \left[ A(t) - B(t)B^*(t)P(t) \right] x(t) + B(t)v(t), \;\; x(t_0) = x_0, \;\; x(t_1) = x_1
\end{aligned} \tag{5.84}$$

The solution of which is given by the above lemma, i.e., $v_{opt}(t) = -B(t)^* \Phi^*(t_0,t) W^{-1}(t_0,t_1)\left[ \Phi(t_0,t_1)x_1 - x_0 \right]$ (or other state feedback forms obtained) where the state transition matrix and the backward controllability Grammian correspond to the system (5.84) with the "new A matrix" $A(t) - B(t)B^*(t)P(t)$. Once $v_{opt}(t)$ is computed, $u_{opt}(t)$ can be solved for.

Note:

The relationship between $v(t)$ and $u(t)$ is shown in the block diagram below.