Lecture 15: Linear Dynamical Systems

4.3.2 Nonhomogeneous Differential Equations

Consider the nonhomogeneous differential equation:

\[ \dot{x}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0. \]  \hspace{1cm} (4.17)

where \( A, f \) are piecewise continuous.

Recall that the solution to \( Ax = y \) is \( x = A^{-1}y + x_0 \).

The solution is the sum of a particular solution \( x_p \) and a homogeneous solution \( x_h \). To make the analogy with (4.17), write:

\[ [D - A(t)] x(t) = f(t), \quad D \equiv \frac{d}{dt} \]  \hspace{1cm} (4.18)

Clearly, the homogeneous solution of (4.18) is the zero-input response for \( f(t) = 0 \), which is determined after \( x(t_0) = x_0 \) is fixed.

Namely, the homogeneous solution is:

\[ \Phi(t, t_0)x_0. \]  \hspace{1cm} (4.19)

To find the overall solution, let \( z(t) = \Phi(t_0, t)x(t) \).

\[ \dot{z}(t) = \dot{\Phi}(t_0, t)x(t) + \Phi(t_0, t)\dot{x}(t) \]

\[ = \frac{d}{dt} \left[ \Phi^{-1}(t_0, t) \right] x(t) + \Phi(t_0, t)\dot{x}(t) \]

\[ = -\Phi(t, t_0)\Phi^{-1}(t_0, t)x(t) + \Phi(t_0, t)\dot{x}(t) \]  \hspace{1cm} (4.20)

\[ = -A\Phi(t_0, t)x(t) + \Phi(t_0, t)[A(t)x(t) + f(t)] \]

\[ = \Phi(t_0, t)f(t) \]

Integrating \( \dot{z}(t) \), we obtain

\[ z(t) = z(t_0) + \int_{t_0}^{t} \Phi(t_0, \sigma)f(\sigma)d\sigma, \]  \hspace{1cm} (4.21)
and from \( x(t) = \Phi(t,t_0)z(t) \), we get:

\[
x(t) = \Phi(t,t_0)x(t_0) + \Phi(t,t_0)\int_{t_0}^{t} \Phi(t_0,\sigma)f(\sigma)d\sigma
\]

\[
= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\sigma)f(\sigma)d\sigma
\]

(4.22)

**Theorem:** (variation of constants formula)

Let \( \Phi(t,t_0) \) be the transition matrix for \( \dot{x}(t) = A(t)x(t) \). Then, the unique solution of

\[
\dot{x}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0
\]

(4.23)

is given by

\[
x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\sigma)f(\sigma)d\sigma.
\]

(4.24)

**Proof:**

Follows from the above construction.

**Corollary:**

The solution of the nonhomogeneous differential equation

\[
\dot{x}(t) = Ax(t) + f(t), \quad x(t_0) = x_0
\]

(4.25)

is given by:

\[
x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\sigma)}f(\sigma)d\sigma
\]

(4.26)

**Theorem:** (stability)

All solutions of the linear time-invariant differential equation \( \dot{x}(t) = Ax(t) \) approach zero as \( t \to \infty \), if all the zeros of \( \det(sI - A) \) lie in the open left half-plane \( \Re\{s\} < 0 \).

All solutions are bounded for positive \( t \), if:

(i) all the zeros of \( \det(sI - A) \) lie in the closed left half-plane \( \Re\{s\} \leq 0 \), and
(ii) for \( s_i \) a zero with vanishing real part and multiplicity \( \sigma_i > 1 \), the following should hold:

\[
\frac{d^{\sigma_i-1-k}}{ds^{\sigma_i-1-k}} \left[ (s-s_i)^{\sigma_i} (sI-A)^{-1} \right]_{s=s_i} = 0, \quad k = 1, 2, \ldots, \sigma_i - 1
\]

**Proof:**

The solution of \( \dot{x}(t) = Ax(t), \ x(t_0) = x_0 \) is obtained from the Laplace transform:

\[
X(s) = (sI - A)^{-1} x_0
\]  

(4.27)

By the inverse Laplace transform formula:

\[
x(t) = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} \left[ \det(sI-A) \right]^{-1} \left[ \text{Adj}(sI-A) \right] e^{st} ds x_0
\]  

(4.28)

where \( \sigma \) is in the region of convergence. Since \( x(t) = e^{At} x_0 \) (assuming \( t \geq t_0 = 0 \)), we have

\[
e^{At} = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} \left[ \det(sI-A) \right]^{-1} \left[ \text{Adj}(sI-A) \right] e^{st} ds
\]  

(4.29)

Note that \( \text{Adj}(sI-A) \) is a matrix of polynomials and that the poles of the integrand are the zeros of \( \det(sI-A) \), e.g., the eigenvalues of \( A \).

Let \( s_1, s_2, \ldots, s_m \) be the eigenvalues of \( A \) and suppose \( s_i \) is repeated \( \sigma_i \) times. From the Cauchy integral formula,

\[
e^{At} = \sum_{i=1}^{m} \sum_{k=0}^{\sigma_i} \frac{1}{k!} \frac{1}{(\sigma_i-1-k)!} \frac{d^{\sigma_i-1-k}}{ds^{\sigma_i-1-k}} \left[ (s-s_i)^{\sigma_i} (sI-A)^{-1} \right]_{s=s_i} t^k e^{st} \].
\]  

(4.30)

Hence, \( e^{At} \) is the sum of constant matrix coefficients times \( t^k e^{st} \).

If \( \text{Re}\{s_i\} < 0, \ \forall i \), then \( e^{At} \xrightarrow{t \to +\infty} 0 \).

If anyone of the eigenvalues of \( A \) has a zero real part, then there is either a term \( A_i e^{js_i t} \) (for a distinct eigenvalue) which does not go to zero, but remains bounded for all times, or terms \( A_{i,k} t^k e^{js_i t} \) (for a repeated eigenvalue) which grow unbounded. Hence, we require the matrix coefficients \( A_{i,k} \) to be null for \( e^{At} \) to remain bounded.
4.3.3 Linear Matrix Equations

Next, we extend nonhomogeneous linear differential equations to nonhomogeneous linear matrix differential equations:

\[ \dot{X}(t) = A_1(t)X(t) + X(t)A_2(t) + F(t), \quad X(t_0) = X_0, \quad (4.31) \]

where \( X(t) \in \mathbb{R}^{n \times n} \), \( t \in T \), and \( A_1, A_2, F \) have piecewise continuous entries.

These matrix differential equations arise naturally in optimal control and filtering problems when the cost function is quadratic (in this case \( F(t) \) is quadratic in \( X(t) \)).

**Theorem:** (matrix variation of constants formula)

Let \( \Phi_1(t,t_0) \) be the transition matrix for \( \dot{x}(t) = A_1(t)x(t) \) and \( \Phi_2(t,t_0) \) be the transition matrix for \( \dot{x}(t) = A_2^T(t)x(t) \). Then, a solution of the linear matrix differential equation

\[ \dot{X}(t) = A_1(t)X(t) + X(t)A_2(t) + F(t), \quad X(t_0) = X_0 \quad (4.32) \]

is given by:

\[ X(t) = \Phi_1(t,t_0)X(t_0)\Phi_2(t,t_0) + \int_{t_0}^{t} \Phi_1(t,\sigma)F(\sigma)\Phi_2^T(\sigma)\left. d\sigma \right. . \quad (4.33) \]

Moreover, \( X(t) \) is unique.

**Proof:**

First, we verify that \( X(t) \) verifies the differential equation:
Next, we address the existence and uniqueness of the solution. Write the linear matrix differential equation in a vector form, where the state vector is composed of all $n^2$ entries of $X(t)$:

\[
x(t) := \begin{bmatrix} X_{11}(t) & \cdots & X_{11}(t) \end{bmatrix}^{\text{T}}.
\]

Then, the resulting system is linear nonhomogeneous, and since $A_1, A_2, F$ have piecewise continuous entries, by the extended Cauchy-Peano theorem, the solution exists and is unique.

\subsection{4.3.4 Impulse Response of a Linear Time-Varying System}

Recall that the most general description of the class of linear systems is obtained via the impulse response. The impulse response of a linear time-varying (LTV) system is defined as the output of the system at time $t$, when the input is an impulse applied at time $\tau$.

\[
u(t) = \delta(t - \tau) \quad \xrightarrow{S} \quad y(t) = (Su)(t) = h(t, \tau)
\]
For an LTI system, it does not matter at what time the impulse is applied: the response impulse response will simply be the \( h(t, \tau) = h(t - \tau) \), so that \( h(t) \) completely and uniquely characterizes an LTI system.

**Definition: Integral representation of an LTV operator \( T \)**

Some notation first:

\[
C(\mathbb{R}; \mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \right\}
\]

(4.35)

We say that \( T \), a linear time-varying operator \( T : C(\mathbb{R}; \mathbb{R}) \to C(\mathbb{R}; \mathbb{R}) \) admits an integral representation if there exists an integrable function (in the Riemann or Lebesgue sense)

\[
g_T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}
\]

called the *kernel* such that for any \( f \in C(\mathbb{R}; \mathbb{R}) \),

\[
(\mathcal{I}f)(t) = \int_{-\infty}^{+\infty} f(\tau) g_T(t, \tau) d\tau.
\]

(4.36)

**Lemma:**

An LTV system that admits an integral representation has an impulse response equal to the kernel of the integral representation.

**Proof:**

Follows from the sampling property of the impulse.

**Note:**

Let \( T \) be a linear time-varying operator \( T : C(\mathbb{R}; \mathbb{R}^m) \to C(\mathbb{R}; \mathbb{R}^p) \) (a multivariable LTV system), and suppose \( T \) admits an integral representation:

\[
y(t) = (T u)(t) = \int_{-\infty}^{+\infty} G_T(t, \tau) u(\tau) d\tau
\]

(4.37)

where the kernel \( G_T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{p \times m} \) is integrable and \( u \in \mathcal{U}, \; y \in \mathcal{Y} \). If we assume that \( T \) is a causal operator, then \( G_T(t, \tau) = 0, \; \forall t < \tau \) and,
If the system is at rest at time $t_0$,

$$y(t) = \int_{t_0}^{t} G_T(t, \tau) u(\tau) d\tau$$

(4.39)

Suppose that the LTV system is actually a state-space system of the form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = 0$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

(4.40)

Then, from (4.24), the input/output relationship is given by:

$$y(t) = C(t)\int_{t_0}^{t} \Phi(t, \sigma) B(\sigma) u(\sigma) d\sigma + D(t)u(t)$$

(4.41)

From this result and (4.39), we find that the kernel is given by:

$$G_T(t, \tau) = \begin{cases} 
C(t)\Phi(t, \tau) B(\tau) + D(t)\delta(t-\tau), & t \geq \tau \\
0, & t < \tau 
\end{cases}$$

(4.42)

and finally, for LTI state-space systems, the kernel (or impulse response) is given by:

$$G_T(t, \tau) = G_T(t - \tau) = \begin{cases} 
Ce^{A(t-\tau)} B + D\delta(t-\tau), & t \geq \tau \\
0, & t < \tau 
\end{cases}$$

(4.43)

or,

$$G_T(t) = \begin{cases} 
Ce^{At} B + D\delta(t), & t \geq 0 \\
0, & t < 0 
\end{cases}$$

(4.44)