Lecture 1: Linear Vector Spaces: Definition and Properties

1 Linear Vector Spaces

This chapter reviews some basic concepts of linear vector spaces and linear algebra. Some motivation to study linear spaces is provided by the following typical problems which occur in engineering:

1. Solutions of \( Ax = y \), where \( A \) is a known \( m \times n \) matrix and \( y \) is a known \( m \)-dimensional vector, but \( x \) is an unknown \( n \)-dimensional vector.
   - Existence of a solution?
   - How many solutions? What are they?

2. Eigenvalues and eigenvectors of the square \( A \) matrix

3. Similarity transformations: \( \hat{A} = PAP^{-1} \)

4. Computation of \( A^k \), \( k = 100 \)

5. Solutions of \( \dot{x}(t) = Ax(t) \) with \( A \) square and the initial state given as \( x(0) = x_0 \)

1.1 Linear Spaces Over a Field

1.1.1 Example of linear space over a field

Cartesian \( n \)-space \( \mathbb{R}^n \)

\( \mathbb{R}^n \) denotes the set of all objects of the form
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
  x_1 & x_2 & \cdots & x_n
\end{bmatrix}
\]
with \( x_i \in \mathbb{R} \), \( i = 1, \ldots, n \).

If for \( \mathbb{R}^n \) we define the sum "+" and multiplication "." operators by
\[
\begin{bmatrix}
  x_1 & x_2 & \cdots & x_n \\
\end{bmatrix} + \begin{bmatrix}
  y_1 & y_2 & \cdots & y_n \\
\end{bmatrix} = \begin{bmatrix}
  x_1 + y_1 & x_2 + y_2 & \cdots & x_n + y_n \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
  x_1 & x_2 & \cdots & x_n \\
\end{bmatrix} \cdot \alpha = \begin{bmatrix}
  \alpha x_1 & \alpha x_2 & \cdots & \alpha x_n \\
\end{bmatrix}, \quad \forall \alpha \in \mathbb{R}
\]
then \( \mathbb{R}^n \) is an example of a finite-dimensional vector space over the field \( \mathbb{R} \).

1.1.2 Definition of a field

Given:

- A set \( F \) of elements \( \alpha, \beta, \gamma, \ldots \)
- "." multiplication by elements of \( F \)
• "+" summation of elements of $\mathcal{F}$

The set $\mathcal{F}$ is called a field if the following conditions hold:

1. Given any pair of elements $(\alpha, \beta)$ in $\mathcal{F}$, their sum $\alpha + \beta$ is an element of $\mathcal{F}$ and their product $\alpha \cdot \beta$ is also an element of $\mathcal{F}$ [closed under ",", and "+"]

2. $\alpha + \beta = \beta + \alpha$, $\alpha \cdot \beta = \beta \cdot \alpha, \forall \alpha, \beta \in \mathcal{F}$ [commutativity]

3. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma), (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma), \forall \alpha, \beta, \gamma \in \mathcal{F}$ [associativity]

4. $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma), \forall \alpha, \beta, \gamma \in \mathcal{F}$ [multiplication is distributive with respect to addition]

5. There exists elements in $\mathcal{F}$ denoted by 1 and 0 such that $\alpha + 0 = \alpha, \alpha \cdot 1 = \alpha, \forall \alpha \in \mathcal{F}$ [additive and multiplicative identities]

6. To each $\alpha \in \mathcal{F}$, there exists an element in $\mathcal{F}$ denoted by $-\alpha$ such that $\alpha + (-\alpha) = 0, \forall \alpha \in \mathcal{F}$ [additive inverse]

7. To each element $\alpha \neq 0 \in \mathcal{F}$ there is an element in $\mathcal{F}$ denoted by $\alpha^{-1}$, such that $\alpha \cdot \alpha^{-1} = 1, \forall \in \mathcal{F}$ [multiplicative inverse]

**Examples:**

(a) Field of scalars defined by usual operations "+" and "." on reals, complex, integers,

(b) $\mathbb{R}(s)$: rational functions of the complex variable $s$ with real coefficients,

(c) Binary numbers $\{0,1\}$ with operations defined as:

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(d) Set of all $2 \times 2$ matrices of the form $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ where $x, y \in \mathbb{R}$,

(e) $\mathbb{R}_+$ (positive numbers) does not satisfy the conditions of a field.

### 1.1.3 Definition of a linear vector space $(\mathcal{X}, \mathcal{F})$

Given:

- a field $\mathcal{F}$
• a set of elements \( \mathcal{X} \)
• \( \cdot \) is a multiplication by scalar \( \alpha \cdot x, \alpha \in \mathcal{F}, x \in \mathcal{X} \)
• \( + \) is a sum in \( \mathcal{X}, x + y, x, y \in \mathcal{X} \)

The set \( (\mathcal{X}, \cdot, +) \) is a linear space over \( \mathcal{F} \) (denoted by \( (\mathcal{X}, \mathcal{F}) \)) if the following conditions hold:

1. To every \( x_i \) and \( x_j \) in \( \mathcal{X} \), there is a vector \( x_i + x_j \in \mathcal{X} \) called the sum of \( x_i \) and \( x_j \)
2. \( x + y = y + x, \forall x, y \in \mathcal{X} \). [commutativity]
3. \( (x + y) + z = x + (y + z), \forall x, y, z \in \mathcal{X} \) [associativity]
4. \( \mathcal{X} \) contains an element denoted by \( \theta \) (called the zero vector) such that \( x + \theta = x, \forall x \in \mathcal{X} \). [additive identity]
5. To every \( x \in \mathcal{X} \), there is a unique element denoted by \( -x \in \mathcal{X} \) such that \( x + (-x) = \theta, \forall x \in \mathcal{X} \). [additive inverse]
6. For each \( \alpha \in \mathcal{F} \) and each \( x \in \mathcal{X} \), there is an element \( \alpha x \in \mathcal{X} \) called the scalar product of \( \alpha \) and \( x \). [scalar multiplication]
7. \( (\alpha \beta) x = \alpha (\beta x), \forall x \in \mathcal{X}, \alpha, \beta \in \mathcal{F} \). [scalar multiplication is associative]
8. \( \alpha (x + y) = \alpha x + \alpha y, \forall x, y \in \mathcal{X}, \alpha \in \mathcal{F} \). [scalar multiplication is distributive with respect to vector addition]
9. \( (\alpha + \beta) x = \alpha x + \beta x, \forall x \in \mathcal{X}, \alpha, \beta \in \mathcal{F} \). [scalar multiplication is distributive with respect to scalar addition]
10. With 1 being the multiplicative identity in \( \mathcal{F} \), \( 1 x = x, \forall x \in \mathcal{X} \).

Examples:

(a) \( (\mathbb{R}, \mathbb{R}), (\mathbb{C}, \mathbb{C}), (\mathbb{C}, \mathbb{R}), (\mathbb{R}(s), \mathbb{R}(s)), (\mathbb{R}(s), \mathbb{R}) \),

however \( (\mathbb{R}, \mathbb{C}), (\mathbb{R}, \mathbb{R}(s)) \) are not linear spaces.

(b) \( (\mathbb{R}^n, \mathbb{R}), \mathbb{R}^n = \{ (\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in \mathbb{R}, \forall i = 1, 2, \ldots, n \} \) or

\[
\mathbb{R}^n \text{ denotes all n-tuples of scalars of the form } x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{bmatrix} \text{ where the first subscript denotes various}
\]

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components of $x_i$ and the second subscript denotes different vectors in $\mathbb{R}^n$.

Here, the operations are defined as:

Sum: $x_i + x_j = \begin{bmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{bmatrix}$, product: $\alpha x_i = \begin{bmatrix} \alpha x_{1i} \\ \alpha x_{2j} \\ \vdots \\ \alpha x_{ni} \end{bmatrix}$, $\alpha \in \mathbb{R}$

(c) $\left( \mathbb{C}^n, \mathbb{C} \right), \left( \mathbb{R}^n(\alpha), \mathbb{R}(\alpha) \right)$

(d) $\mathbb{R}_n[\alpha]$: set of polynomials of degree less than $n$ with real coefficients $\sum_{i=0}^{n-1} \alpha_i s^i$

Let

Sum: $\sum_{i=0}^{n-1} \alpha_i s^i + \sum_{i=0}^{n-1} \beta_i s^i = \sum_{i=0}^{n-1} (\alpha_i + \beta_i) s^i$

Product: $\alpha \left( \sum_{i=0}^{n-1} \alpha_i s^i \right) = \sum_{i=0}^{n-1} (\alpha \alpha_i) s^i, \alpha \in \mathbb{R}$

Then, $\left( \mathbb{R}_n[\alpha], \mathbb{R} \right)$ is a linear space. But, $\left( \mathbb{R}_n[\alpha], \mathbb{R}(\alpha) \right)$ is not a linear space.

(e) $C[a,b]$ Function Space (space of continuous functions)

$\mathcal{X} = \left\{ f : [a,b] \rightarrow \mathbb{R} : f$ are continuous signals on a continuous interval $[a,b]$ in $\mathbb{R} \right\}$

$\mathcal{F}$

"*" is defined pointwise by $(\alpha f)(t) = \alpha \cdot f(t), \forall t \in (-\infty, \infty), f \in \mathcal{X}, \alpha \in \mathbb{R}$.

"+" is defined pointwise by $(f_1 + f_2)(t) = f_1(t) + f_2(t), \forall t \in (-\infty, \infty), f_1, f_2 \in \mathcal{X}$.

"$\theta$" is $\theta(t) = 0, \forall t \in (-\infty, \infty)$

"$-f$" is $(-f)(t) = -f(t), \forall t \in (-\infty, \infty)$

Conditions 1 to 10 hold; hence this class of functions is a linear space over $\mathbb{R}$. It is called a function space and is denoted by $C[a,b]$.

### 1.1.4 Definition of a subspace $(\mathcal{Y}, \mathcal{F})$ of a linear space $(\mathcal{X}, \mathcal{F})$

Given:

- a linear space $(\mathcal{X}, \mathcal{F})$
- $\mathcal{Y}$ a subset of $\mathcal{X}$

Then $(\mathcal{Y}, \mathcal{F})$ is called a subspace of $(\mathcal{X}, \mathcal{F})$ if, under the operations of $(\mathcal{X}, \mathcal{F}), (\mathcal{Y}, \mathcal{F})$ is a linear space.
Notes:

1) From the definition, it follows that
\[ \forall y_1, y_2 \in Y \quad \forall \alpha_1, \alpha_2 \in F \quad \Rightarrow \quad \{ \alpha_1 y_1 + \alpha_2 y_2 \in Y \} \]

2) Conditions 2, 3, 7, 8, 9 and 10 are satisfied because \( Y \) is a subset of \( X \) and both are over the same field. Hence, we only need to verify conditions 1, 4, 5 and 6.
Conditions 1, 4, 5 and 6 hold if
\[ \{ \alpha y \in Y \} \]

Examples:

(a) \((X, F)\) with \( F = \mathbb{R}, Y = \{(x, x, z) : x, z \in \mathbb{R}\} \) is a subspace of \((\mathbb{R}^3, \mathbb{R})\).

Verify; recall
"+": \((x, x, z) + (x', x', z') = (x + x', x + x', z + z')\)
"\(\times\)": \(\alpha (x, x, z) = (\alpha x, \alpha x, \alpha z)\)

\(Y\) is closed under addition (e.g., 1 holds)
\(Y\) is closed under scalar multiplication (e.g., 6 holds)
There is an additive inverse (e.g., 5 holds)
\[ \forall (x, x, z) \in Y, \quad (x, x, z) + (-x, -x, -z) = \theta, \quad (-x, -x, -z) \in Y, \quad \theta \in Y \]
with \( \theta = (0, 0, 0) \)

So, \((x, x, y) + \theta = (x, x, y), \forall x, y \in \mathbb{R}\), e.g., 4 holds
Alternatively, check if \( \forall y_1, y_2 \in Y, \quad \alpha_1, \alpha_2 \in \mathbb{R}, \quad \alpha_1 y_1 + \alpha_2 y_2 \in Y \).
Since
\[ \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 (x, x, z) + \alpha_2 (x', x', z') \]
\[ = (\alpha_1 x + \alpha_2 x', \alpha_1 x + \alpha_2 x', \alpha_1 z + \alpha_2 z') \]
\[ = (\tilde{x}, \tilde{x}, \tilde{z}) \in Y \]
then \((Y, F)\) is a subspace of \((\mathbb{R}^3, \mathbb{R})\).

(b) \( \mathbb{R}^2 \)

\[ Y = \{(x, \alpha x) : x \in \mathbb{R}, \alpha \in \mathbb{R}\} \]
subspace of \( \mathbb{R}^2 \)
(c) Let $\mathcal{Y} := \{ Ax : x \in \mathbb{R}^n, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, A \text{ is an } m \times n \text{ matrix with real entries} \}$

Then $(\mathcal{Y}, \mathbb{R})$ is a subspace of $(\mathbb{R}^m, \mathbb{R})$, called the image or range of $A$.

Check:
First $\mathcal{Y} \subseteq \mathbb{R}^m$. Each $y \in \mathcal{Y}$ is of the form $Ax$, for some $x \in \mathbb{R}^n$

Conditions 1,6: For $y_1, y_2 \in \mathcal{Y}, \alpha_1, \alpha_2 \in \mathbb{R}$,

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) \in \mathcal{Y}, \text{ where } (x_1 + x_2) \in \mathbb{R}^n$$

$$\alpha_1 y_1 = \alpha(Ax) = A(\alpha x) \in \mathcal{Y}, \text{ where } (\alpha x) \in \mathbb{R}^n$$

Condition 4: $A\theta_n = \theta_m \Rightarrow \theta_m \in \mathcal{Y}, \theta_n \in \mathbb{R}^n$

Condition 5: $Ax + A(-x) = A(x + (-x)) = A\theta_n = \theta_m$ where $(x + (-x)) \in \mathbb{R}^n$

(d) Let $\mathcal{Y} := \{ x \in \mathbb{R}^n : Ax = \theta_m, A \in \mathbb{R}^{m \times n} \}$

Then $(\mathcal{Y}, \mathbb{R})$ is a subspace of $(\mathbb{R}^n, \mathbb{R})$ called the kernel or nullspace of $A$.

Check:
First $\mathcal{Y} \subseteq \mathbb{R}^n$ because $x \in \mathbb{R}^n$.

Condition 4: $\theta_n \in \mathcal{Y}$ because $A\theta_n = \theta_m$

Conditions 1,6: For $x_1, x_2 \in \mathcal{Y}, \alpha \in \mathbb{R}$

$$A(x_1 + x_2) = Ax_1 + Ax_2 = \theta_m + \theta_m = \theta_m$$

$$A(\alpha x) = \alpha Ax = \alpha \theta_m = \theta_m$$

Condition 5: For $x \in \mathcal{Y}$ then $(-1)x \in \mathcal{Y}$ from 6.

(e) Is $(\mathcal{Y}, F), F \in \mathbb{R}, \mathcal{Y} = \{(x, z) \in \mathbb{R}^2 : x^2 + z^2 \leq 1 \}$ a subspace of $(\mathbb{R}^2, \mathbb{R})$? No!

\[ \mathbb{R}^2 \]

$\mathcal{Y}$

$1$

$1$
(f) \( D[a, b] \) function space of differentiable functions. \((D[a, b], \mathbb{R})\) is a subspace of \((C[a, b], \mathbb{R})\), e.g., \( \theta(t) \in C[a, b] \) and \( \theta'(t) = \theta(t) \in D[a, b] \)

If \( f_1 + f_2 \in D[a, b] \) then \( (f_1 + f_2)' = f_1' + f_2' \in D[a, b] \)...

Note:

Let \( \mathcal{Y} \) be a subset of a vector space \((\mathcal{X}, \mathcal{F})\). Then, \((\mathcal{Y}, \mathcal{F})\) is a subspace of \((\mathcal{X}, \mathcal{F})\) if it satisfies the following conditions:

1. \( \theta \in \mathcal{Y} \), where \( \theta \) is the zero vector of \((\mathcal{X}, \mathcal{F})\)
2. if \( y_1, y_2 \in \mathcal{Y} \) then \( y_1 + y_2 \in \mathcal{Y} \)
3. if \( y \in \mathcal{Y} \) then \( \alpha y \in \mathcal{Y}, \forall \alpha \in \mathcal{F} \)

Proposition:

Let \((\mathcal{Y}_1, \mathcal{F}), (\mathcal{Y}_2, \mathcal{F})\) be subspaces of \((\mathcal{X}, \mathcal{F})\). Then,

a) \( \mathcal{Y}_1 \cap \mathcal{Y}_2 \) over \( \mathcal{F} \) is a linear subspace of \((\mathcal{X}, \mathcal{F})\)

b) \( \mathcal{Y}_1 \cap \mathcal{Y}_2 \) over \( \mathcal{F} \) is a linear subspace of \((\mathcal{Y}_1, \mathcal{F})\) and of \((\mathcal{Y}_2, \mathcal{F})\)

c) If \((\mathcal{Y}_1, \mathcal{F})\) is a linear subspace of \((\mathcal{Y}_2, \mathcal{F})\) then \(((\mathcal{Y}_1 \cap \mathcal{Y}_2)), \mathcal{F})\) is exactly \((\mathcal{Y}_1, \mathcal{F})\).

Partial Proof of (a):

\( \mathcal{Y}_1 \cap \mathcal{Y}_2 = \{ y : y \in \mathcal{Y}_1 \text{ and } y \in \mathcal{Y}_2 \} \)

if \( y \in \mathcal{Y}_1 \cap \mathcal{Y}_2 \) then \( y \in \mathcal{Y}_1 \) and \( y \in \mathcal{Y}_2 \)
if \( y_1, y_2 \in \mathcal{Y}_1 \cap \mathcal{Y}_2 \) then \( y_1, y_2 \in \mathcal{Y}_1 \) and \( y_1, y_2 \in \mathcal{Y}_2 \)
\( y_1 + y_2 \in \mathcal{Y}_1 \) and \( y_1 + y_2 \in \mathcal{Y}_2 \) [\( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are linear spaces]
\( y_1 + y_2 \in \mathcal{Y}_1 \cap \mathcal{Y}_2 \)